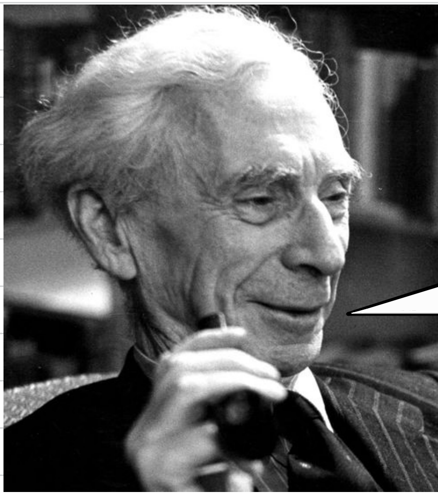
① Intro: Math Without Choice

Obligatory example

What is the difference between doing math with & without the axiom of choice?



You can pick one shoe from each of infinitely many pairs of shoes

 . . .

always choose the left shoe

But you can't do the same for socks

 . . .

identical, so no way to choose

<u>Aside</u> When Russell originally gave this example he was careful to say you probably actually could choose socks by using small physical differences (e.g. mass)

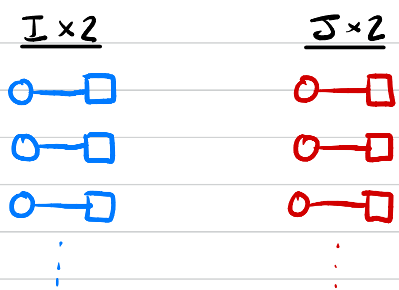
② You Can Divide by 3 Without Choice

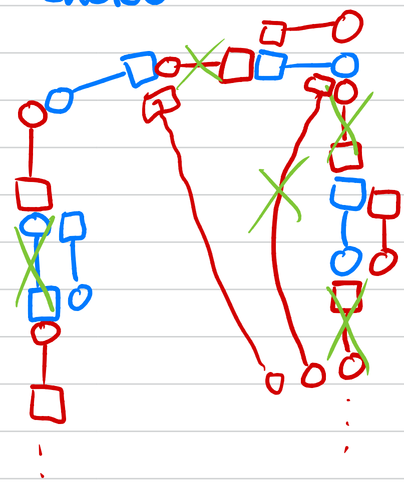
Most facts about cardinal arithmetic are not provable without choice. But surprisingly...

<u>Thm</u> (ZF) If I and J are sets such that $|I \times 2| = |J \times 2|$ then $|I| = |J|$

You can "divide by 2" without choice

<u>proof</u>





① Flip blue
② Swap
③ Erase

✓

$I \hookrightarrow J$

Thm For any $n \in \mathbb{N}$ and any sets I and J,

$$|I \times n| = |J \times n| \Rightarrow |I| = |J|$$

History

1901 Bernstein's thesis
division by 2, claimed proof of division by n
correctness is disputed

1922 Sierpinski
division by 2, tried & failed to divide by 3

1926 Lindenbaum & Tarski
division by n; paper with 144 theorems, no proofs

1949 Tarski
1st published accepted proof of division by n
Tarski said this wasn't the 1926 proof

"Division by Three"

1994 Conway & Doyle
simpler proof (maybe L&T's original proof)

"Division by Four"

2015 Doyle & Qiu
really simple proof

"Pangalactic Division"

③ Can You Divide by 3 Without Choice?

It is questionable whether this theorem is what "division by n" should mean

↪ Is $I \times n$ the right definition of "multiplying I by n"?

Alternate definition: Multiplying I by n means any set of disjoint sets of size n indexed by I

$\{A_i\}_{i \in I}$ $A_i \cap A_j = \emptyset$ $|A_i| = n$ $\forall i \in I$

Note that the proof used the ordering of the fibers

<u>Question</u> Suppose $n \in \mathbb{N}$, I and J are two sets and $\{A_i\}_{i \in I}$, $\{B_j\}_{j \in J}$ are both collections of disjoint sets of size n. Does

$$\left|\bigcup_{i \in I} A_i\right| = \left|\bigcup_{j \in J} B_j\right|$$

imply $|I| = |J|$?

Conway Can Divide By 3,

But I Can't.

④ You Can't Divide by 3 (or even 2!) Without Choice

Def **Shoe division by n** is the following principle:

$$\boxed{\text{If } |I \times n| = |J \times n| \text{ then } |I| = |J|}$$

Def **Sock division by n** is the following principle

> If $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ are both collections of disjoint sets of size n then
> $$|\bigcup A_i| = |\bigcup B_j| \Rightarrow |I| = |J|$$

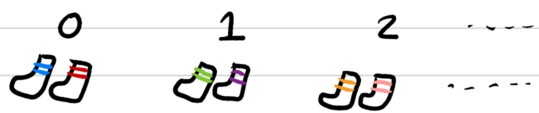
Bernstein, Tarski, Lindenbaum, Conway, Doyle, Qiu: $ZF \vdash$ shoe division by n

Question: $ZF \vdash$ sock division by n?

Answer No!

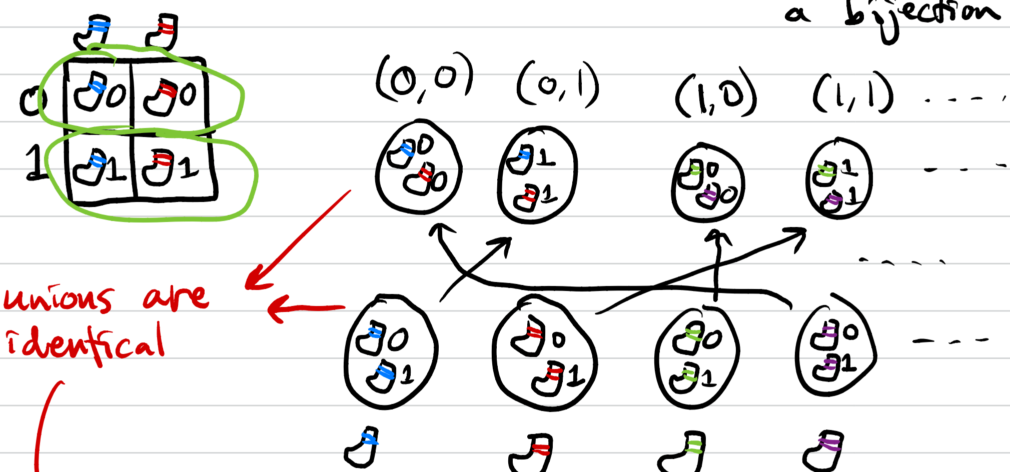
Thm Sock division by 2 is not provable in ZF

proof We will show that if this is possible then we can choose socks for Bertrand Russell's millionaire



$\{A_i\}_{i\in\mathbb{N}}$ $\quad |A_i| = 2$

We will use sock division to find a bijection $\bigcup A_i \to \mathbb{N}\times 2$



$\approx \mathbb{N}\times 2$

$\approx \bigcup_{i\in\mathbb{N}} A_i$

unions are identical

So by sock division, $|\bigcup A_i| = |\mathbb{N}\times 2|$

⑤ Shoes vs. Socks

The difference between shoe division & sock division comes down to two different ways to define "multiplication by n" without choice

Multiplication by n

$|I| \times n = |I \times n|$

not well defined!

Repeated Addition of n

$|I| \times n = |\bigcup A_i|$ where $\{A_i\}_{i \in I}$ is any collection of disjoint sets of size n

It turns out that Russell first used his story about shoes & socks to explain this exact issue!

"Introduction to Mathematical Philosophy" 1919

Another illustration may help to make the point clearer. We know that $2 \times \aleph_0 = \aleph_0$. Hence we might suppose that the sum of $\aleph_0$ pairs must have $\aleph_0$ terms. But this, though we can prove that it is sometimes the case, cannot be proved to happen *always* [page 126] unless we assume the multiplicative axiom. This is illustrated by the millionaire who bought a pair of socks whenever he bought a pair of boots, and never at any other time, and who had such a passion for buying both that at last he had $\aleph_0$ pairs of boots and $\aleph_0$ pairs of socks. The problem is: How many boots had he, and how many socks? One would naturally suppose that he had twice as many boots and twice as many socks as he had pairs of each, and that therefore he had $\aleph_0$ of each, since that number is not increased by doubling. But this is an instance of the difficulty, already noted, of connecting the sum of ν classes each having μ terms with μ×ν. Sometimes this can be done, sometimes it cannot. In our case it can be done with the boots, but not with the socks, except by some very artificial device. The reason for the difference is this: Among boots we can distinguish right and left, and therefore we can make a selection of one out of each pair, namely, we can choose all the right boots or all the left boots; but with socks no such principle of selection suggests itself, and we cannot be sure, unless we assume the multiplicative axiom, that there is any class consisting of one sock out of each pair. Hence the problem.

⑥ Generalization

Any cardinal (really, any set)

Def Shoe division by K is the following principle:

$$\boxed{\text{If } |I \times K| = |J \times K| \text{ then } |I| = |J|}$$

Def Sock division by K is the following principle

$$\boxed{\begin{array}{l}\text{If } \{A_i\}_{i\in I} \text{ and } \{B_j\}_{j\in J} \text{ are both collections of} \\ \text{disjoint sets of size } K \text{ then } |\bigcup A_i| = |\bigcup B_j| \Rightarrow |I| = |J|\end{array}}$$

Def "Multiplication by K equals repeated addition of K" is the following principle:

$$\boxed{\begin{array}{l}\text{If } \{A_i\}_{i\in I} \text{ is a collection of disjoint sets of} \\ \text{size } K \text{ then } |\bigcup A_i| = |I \times K|.\end{array}}$$

Thm Sock division by K = Shoe division by K + Multiplication by K equals repeated addition of K

<u>Thm</u> ZF proves that for any K, sock division by K holds if and only if shoe division by K and "multiplication by K equals repeated addition of K" both hold

Sock division = Shoe division + Multiplication is repeated addition

<u>proof</u> Shoe division + repeated addition $\Rightarrow$ sock division

$\{A_i\}, \{B_j\}$ s.t. $|\bigcup A_i| = |\bigcup B_j|$

$$|I\times K| \overset{\text{mult = repeated addition}}{=} |\bigcup A_i| \underset{\text{assumption}}{=} |\bigcup B_j| \overset{\text{mult = repeated addition}}{=} |J\times K| \overset{\text{shoe division}}{\Rightarrow} |I| = |J|$$

Sock division $\Rightarrow$ Shoe division

Sock division $\Rightarrow$ multiplication = repeated addition

$\{A_i\}_{i\in I}$

$$(\bigcup A_i)\times K = \{(i,a,x) \mid i\in I,\ a\in A_i,\ x\in K\}$$

unions are identical

$\{\{(i,a,x) \mid a\in A_i\}\}_{i\in I, x\in K}$ ← indexed by $I\times K$

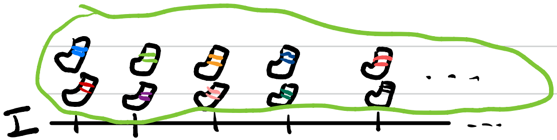
$\{\{(i,a,x) \mid x\in K\}\}_{i\in I, a\in A_i}$ ← indexed by $\bigcup A_i$

$\Rightarrow$ $|I\times K| = |\bigcup A_i|$ ← sock division

⑦ A Question

How powerful is sock division?

Def An n-sock bundle is a collection $\{A_i\}_{i\in I}$ of disjoint sets of size n



I = base space

$\bigcup A_i$ = total space

$\prod A_i$ = global sections of $\{A_i\}_{i\in I}$

We just saw that sock division by n implies that the total spaces of any two n-sock bundles over I are in bijection

Def The trivial n-sock bundle over I is $I \times n$

↳ a.k.a. n-shoe bundle

Def An isomorphism of sock-bundles $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ is a bijection $f : \bigcup A_i \to \bigcup B_i$ such that $f(A_i) = B_i$

Def An n-sock bundle $\{A_i\}_{i\in I}$ can be trivialized if it is isomorphic to the trivial n-sock bundle over I

Question Does sock division imply every sock bundle can be trivialized?