① Reminder

Def A cone of Turing degrees is a set of the form

$\{x \in \mathcal{D}_T \mid x \geq_T y\}$ for some fixed y.

$\mathcal{D}_T$ → Turing degrees

Thm (Martin's cone theorem) Every set of Turing degrees either contains a cone or is disjoint from a cone

↳ Assuming ZF+AD! False in ZFC

Def If $f, g : 2^\omega \to 2^\omega$ are Turing invariant:

$f \leq_M g$ means $f(x) \leq_T g(x)$ on a cone

$f \equiv_M g$ means $f(x) \equiv_T g(x)$ on a cone

iff $f \leq_M g$ and $g \leq_M f$

"Martin order" and "Martin equivalence"

Martin's Conjecture

Assume ZF + AD + DC

1. Every Turing invariant function $f: 2^\omega \to 2^\omega$ is either
   - Martin equivalent to a constant function
     "Constant on a cone"
   - Martin above the identity function
     "$f(x) \geq_T x$ on a cone"

2. Functions which are above the identity are pre-wellordered by the Martin order and the successor in this pre-wellorder is the Turing jump.

   Successor of f is $x \mapsto f(x)'$

② Past Results

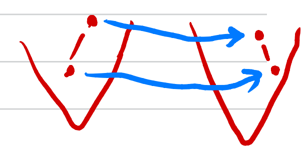
Thm (Slaman–Steel) Martin's conjecture holds for all regressive functions

Thm (Slaman–Steel) Part 2 of Martin's conjecture holds for all order preserving functions which are not above the hyperjump

Def A Turing invariant function $f : 2^\omega \to 2^\omega$ is regressive if $f(x) \leq_T x$ on a cone

Def A Turing invariant function $f : 2^\omega \to 2^\omega$ is order preserving if for all $x, y \in 2^\omega$

$$x \geq_T y \Rightarrow f(x) \geq_T f(y)$$



③ Our Results

Thm Part 1 of Martin's conjecture holds for all measure preserving functions

next week

Thm Every order preserving is either constant on a cone or measure preserving

in two weeks

"order preserving $\Rightarrow$ measure preserving"

Cor Part 1 of Martin's conjecture holds for all order preserving functions

What does "measure preserving" mean?

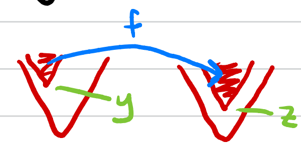
Explaining that is the goal of today's talk

Three Views of Measure Preserving Functions on the Turing Degrees.

④ View #1: Combinatorial Definition

<u>Def</u> A Turing invariant function $f: 2^\omega \to 2^\omega$ is measure preserving if for all z, there is some y such that

$$x \geq_T y \implies f(x) \geq_T z$$



"f eventually gets above z"
"f goes to infinity in the limit"

f sends the cone above y into the cone above z

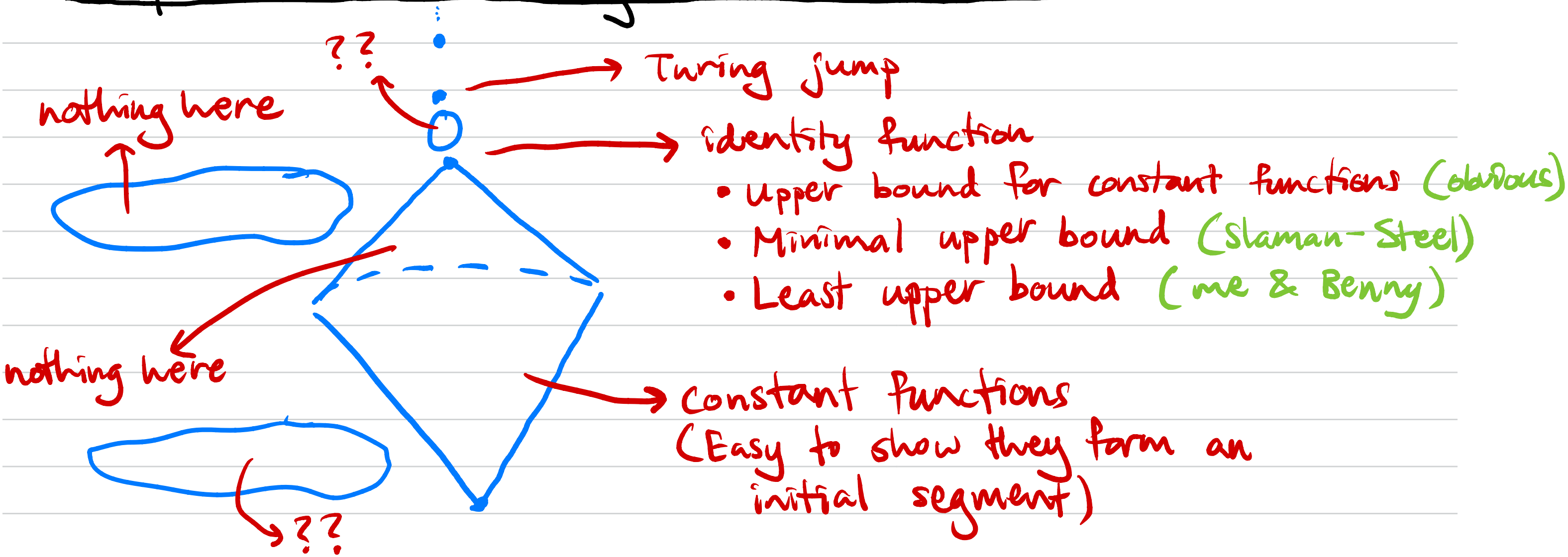
Example: Every function which is above the identity. Surprisingly hard to find others (even in ZFC).

This is the most useful definition for proving things, but maybe not the most interesting definition conceptually

⑤ View #2: Martin Order

<u>Prop</u> A Turing invariant function is measure preserving if and only if it is above every constant function in the Martin order

<u>A picture of the Turing invariant functions:</u>



⑥ View #3: Martin Measure

Some Background

Def An ultrafilter on a set X is a collection $\mathcal{U}$ of subsets of X such that

$A \in \mathcal{U}$ = "A is large"

"filter" [
1. $\emptyset \notin \mathcal{U}$, $X \in \mathcal{U}$
2. $A \in \mathcal{U}$, $A \subseteq B \Rightarrow B \in \mathcal{U}$
3. $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$

"ultra" [
4. $A \subseteq X \Rightarrow$ either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$

Def Martin measure, $\mathcal{U}_M$, is the collection of subsets of the Turing degrees defined by

$$A \in \mathcal{U}_M \iff A \text{ contains a cone}$$

Thm (Martin's cone theorem, restated) $\mathcal{U}_M$ is an ultrafilter

Actually, $\mathcal{U}_M$ is a countably complete ultrafilter

Def If $\mathcal{U}$ is an ultrafilter on a set X and $f: X \to Y$ then the pushforward of $\mathcal{U}$ along f, written $f_*(\mathcal{U})$, is the ultrafilter on Y defined by

$$A \in f_*(\mathcal{U}) \iff f^{-1}(A) \in \mathcal{U}$$

Example If $f: X \to Y$ is constant on a set in $\mathcal{U}$ then $f_*(\mathcal{U})$ is a principal ultrafilter

Example The pushforward of Martin measure along the function $x \longmapsto \omega_1^x$ gives a countably complete ultrafilter on ω_1

Hence $AD \Rightarrow \omega_1$ is measurable!

Def A function $f: X \to X$ is measure preserving for an ultrafilter $\mathcal{U}$ on X if

$$f_*(\mathcal{U}) = \mathcal{U}$$

Prop If $f : 2^\omega \to 2^\omega$ is a Turing invariant function then f is measure preserving if and only if the function on the Turing degrees induced by f preserves Martin measure

proof:

($\Rightarrow$) Given z, $\exists y$ $f(\mathrm{Cone}(y)) \subseteq \mathrm{Cone}(z)$

So $\mathrm{Cone}(y) \subseteq f^{-1}(\mathrm{Cone}(z))$.

And $f^{-1}(\mathcal{D}_T \setminus \mathrm{Cone}(z))$ disjoint from $\mathrm{Cone}(y)$

($\Leftarrow$) Given z, Know $f^{-1}(\mathrm{Cone}(z))$ contains a cone, say $\mathrm{Cone}(y)$. So $f(\mathrm{Cone}(y)) \subseteq \mathrm{Cone}(z)$

⑦ Application: Rudin-Keisler order

Some more background

Def Suppose $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on X

$\mathcal{U} \leq_{RK} \mathcal{V}$ means there is $f : X \to X$ s.t. $f_*(\mathcal{V}) = \mathcal{U}$ ($\mathcal{V} \to \mathcal{U}$)

$\mathcal{U} \equiv_{RK} \mathcal{V}$ means $\mathcal{U} \leq_{RK} \mathcal{V}$ and $\mathcal{V} \leq_{RK} \mathcal{U}$

↳ Not the usual definition!

This gives a quasi order on ultrafilters on X

Example If $\mathcal{U}$ is a principal ultrafilter concentrating at $a \in X$ then $\mathcal{U}$ is $\leq_{RK}$ minimal

$f_*(\mathcal{V}) = \mathcal{U}$ where $f(x) = a$ for all x

Example $\mathcal{U}$ is $\leq_{RK}$-minimal among nonprincipal ultrafilters on ω $\Leftrightarrow$ $\mathcal{U}$ is Ramsey

Thm (Part 1 of Martin's conjecture for measure preserving functions, restated)

Part 1 of Martin's conjecture $\iff$ If $\mathcal{U}$ is a nonprincipal ultrafilter on the Turing degrees $\mathcal{U} \leq_{RK} \mathcal{U}_M \Rightarrow \mathcal{U} = \mathcal{U}_M$

Assuming $AD_{\mathbb{R}}$ or AD^+

- $\mathcal{U}_M$ is minimal
- Nothing else is $\equiv_{RK}$ to it

proof: $f : \mathcal{D}_T \to \mathcal{D}_T$

$f_*(\mathcal{U}_M) = \mathcal{U}_M \iff f$ is measure preserving

$\iff f \geq_M \mathrm{id}$

Caveat: Need more than AD to go from $f : \mathcal{D}_T \to \mathcal{D}_T$ to Turing invariant function on 2^ω

This suggests: Try to prove part 1 of Martin's conjecture by looking at $\leq_{RK}$ on the Turing degrees

- Use ideas from set theory
- Just prove $\mathcal{U}_M$ is $\leq_{RK}$ minimal or just prove nothing is $\equiv_{RK}$ to it
- Look at specific ultrafilters on $\mathcal{D}_T$ and try to show they are not below $\mathcal{U}_M$

⑧ The Lebesgue Ultrafilter

Thm (Kolmogorov 0-1 Law) If $A \subseteq 2^\omega$ is closed under tail equivalence then either

if x & y agree on all but finitely many places then $x \in A \Leftrightarrow y \in A$

Lebesgue measure ← $\mu(A) = 0$ or $\mu(A) = 1$

→ A closed under rational shifts ⇒ measure is 0 or 1

Thm ZF+AD ⇒ Every set is Lebesgue measurable

Cor ZF+AD ⇒ Lebesgue measure is an ultrafilter on the Turing degrees! Let's call it $\mathcal{U}_L$.

Question: Is $\mathcal{U}_L \leq_{RK} \mathcal{U}_M$?

What we can show: $\mathcal{U}_M \not\leq_{RK} \mathcal{U}_L$ ($\Rightarrow \mathcal{U}_M \not\equiv_{RK} \mathcal{U}_L$)

Thm $\mathcal{U}_M \not\leq_{RK} \mathcal{U}_L$

proof: Suppose $\mathcal{U}_M \leq_{RK} \mathcal{U}_L$

So there is a function $f : \mathfrak{D}_T \to \mathfrak{D}_T$ s.t. $f_*(\mathcal{U}_L) = \mathcal{U}_M$

Can view it as $f : 2^\omega \to \mathfrak{D}_T \to \omega_1$

So $x \longmapsto \omega_1^{f(x)}$ shows Lebesgue measure pushes forward to a nonprincipal ultrafilter on ω_1

But every map $2^\omega \to \omega_1$ is constant on a set of positive measure!

Otherwise $\{x \mid \omega_1^{f(x)} \in A\}$ has measure 0 for ctbl A

Look at $B = \{(x,y) \mid \omega_1^{f(x)} < \omega_1^{f(y)}\}$

By Fubini $\displaystyle \int \underbrace{\int B\, dx}_{\mu(\{x \mid \omega_1^{f(x)} < \omega_1^{f(y)}\}) = 0} dy = \int \overbrace{\int B\, dy}^{\mu(\{y \mid \omega_1^{f(x)} < \omega_1^{f(y)}\}) = 1} dx$

Question: Is $\mathcal{U}_L$ strictly below $\mathcal{U}_M$

Thm (Marks) $\mathcal{U}_L <_{RK} \mathcal{U}_M \iff \exists f : 2^\omega \to 2^\omega$ Turing invariant

Assuming $AD_{\mathbb{R}}$

$f(x)$ is x-random for all x

Question: Is $\mathcal{U}_M$ $\leq_{RK}$-maximal among ultrafilters on $\mathcal{D}_T$?