# The Solecki dichotomy and the Posner-Robinson Theorem

Caltech Logic Seminar
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# ① Introduction

Effective descriptive set theory = Dictionary between computability and descriptive set theory

**Examples** ① continuous ≈ computable

② Borel ≈ hyperarithmetic

Backed up by precise theorems

**Thm** For $f : 2^\omega \to 2^\omega$

① $f$ is continuous $\Leftrightarrow$ $f$ is computable relative to an oracle
$\exists a, \Phi$ s.t. for all $x$, $f(x) = \Phi(a, x)$

② $f$ is Borel meas. $\Leftrightarrow$ $f$ is hyp. relative to an oracle

To prove thms in descriptive set thy: relativize to an oracle & then use computability thy

Effective descriptive set theory = Dictionary between computability and descriptive set theory

Another example?

Solecki dichotomy (informally) Every Borel function $f: 2^\omega \to 2^\omega$ is either a ctbl union of continuous functions or at least as complicated as the Turing jump

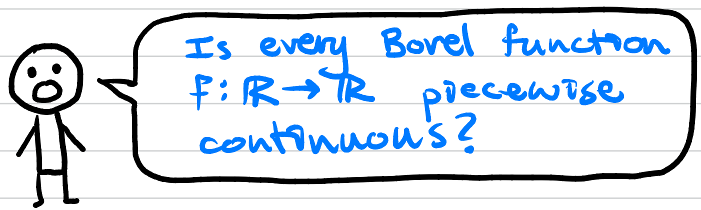
Posner-Robinson thm Every real $x \in 2^\omega$ is either computable or looks like $0'$

Both $\approx$ Every object is computable-(ish) or "jump-like"

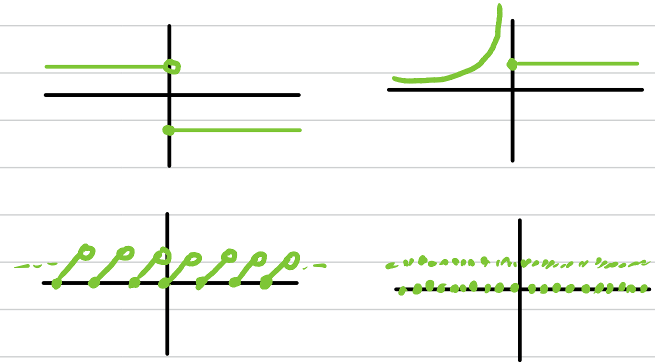
But no obvious way to fit this into the eff. descriptive set thy dictionary

This talk: Posner-Robinson thm can (almost) be used to prove the Solecki dichotomy

## ② The Solecki dichotomy




Nikolai Luzin



<u>Def</u> $f: X \to Y$ is **$\sigma$-continuous** if there is a ctbl partition $\langle A_n \rangle_{n\in\mathbb{N}}$ of $X$ s.t. for each $n$, $f|A_n$ is continuous wrt the subspace topology on $A_n$

<u>Luzin's question (formal version)</u> Is every Borel function $f: \underset{2^\omega}{\cancel{\mathbb{R}}} \to \underset{2^\omega}{\cancel{\mathbb{R}}}$ $\sigma$-continuous?

<u>Answer</u> No! For example, the Turing jump, $J(x) = x'$, is not.

Luzin's question (formal version) Is every Borel function $f: 2^\omega \to 2^\omega$ $\sigma$-continuous?

Answer No! For example, the Turing jump, $J(x) = x'$, is not.

Thm $J$ is not $\sigma$-continuous

pf Suppose it was.

$\Rightarrow$ $\exists \langle A_n \rangle$ s.t. $\forall n$, $J|_{A_n}$ is continuous
$\Rightarrow$ $\exists a_n$, s.t. $\forall x \in A_n$, $x' = J(x) \le_T a_n \oplus x$

Let $a = \bigoplus_n a_n$. Then for all $x$, $x' \le_T a \oplus x$
Take $x \ge_T a$. Then $x' \le_T x$.

Question Are there any other examples?

Answer In a sense, no. $\leftarrow$ Informally, what the Solecki dichotomy says

## 2.1 Solecki dichotomy, formal version

**Def** For $f, g : 2^\omega \to 2^\omega$,

① $f \le_{sW} g$ if $\exists \varphi, \psi : 2^\omega \to 2^\omega$ partial continuous such that $\forall x,\ f(x) = \psi(g(\varphi(x)))$

$$\begin{tikzcd} 2^\omega \arrow[r, "g"] & 2^\omega \arrow[d, "\psi"] \\ 2^\omega \arrow[u, "\varphi"] \arrow[r, "f"] & 2^\omega \end{tikzcd}$$

"strong continuous Weihrauch reducibility"
(also called "continuous reducibility")

② $f \le_W g$ if $\exists \varphi : 2^\omega \to 2^\omega$, $\psi : 2^\omega \times 2^\omega \to 2^\omega$ partial cont. such that $\forall x,\ f(x) = \psi(g(\varphi(x)), x)$

$\psi$ gets to see the original input

"continuous Weihrauch reducibility"

Example $f(x) = \underbrace{1\ldots1}_{n}00\ldots$ s.t. $n = \max_i x(i)$

$f(x) = 111\ldots$ if $\max_i x(i) = \infty$

$f \le_{sW} J$ via $\psi(x') = \left(n \mapsto \begin{cases} 1 & \text{if } x'(e_n) = 0 \\ 0 & \text{else} \end{cases}\right)$

where $e_n$ is s.t. $\Phi_{e_n}(x)\downarrow \iff \exists i\ x(i) > n$

$f \leq_{sw} g$ if $\exists \varphi, \psi : 2^\omega \to 2^\omega$ partial continuous such that $\forall x,\ f(x) = \psi(g(\varphi(x)))$

$f \leq_w g$ if $\exists \varphi : 2^\omega \to 2^\omega$, $\psi : 2^\omega \times 2^\omega \to 2^\omega$ partial cont. such that $\forall x,\ f(x) = \psi(g(\varphi(x)), x)$

Suppose $f : 2^\omega \to 2^\omega$ is Borel

Solecki dichotomy — Either $f$ is $\sigma$-continuous or $J \leq_{sw} f$

Solecki/Zapletal/Pawlikowski-Sabok — Proof by priority argument

Weak Solecki dichotomy — Either $f$ is $\sigma$-continuous or $J \leq_w f$

Proof by Posner-Robinson thm + determinacy

Comment $f \leq_w g$ and $f$ not $\sigma$-cont. $\Rightarrow$ $g$ not $\sigma$-cont.

In a sense, the uncomputability of the halting problem accounts for all instances of Borel functions failing to be $\sigma$-cont.

## ③ The Posner-Robinson theorem

Posner-Robinson thm: Every real $x \in 2^\omega$ is either computable
(informally) or looks like $0'$

**Thm** (Posner-Robinson) For every $x \in 2^\omega$, either $x$ is computable or there is some $g$ such that

$$x \oplus g \geq_T g'$$

$\rightarrow$ relative to $g$, $x$ computes the halting problem

**Relativized** For every $y$ and $x$, either $x \leq_T y$ or there is some $g$ such that

$$x \oplus g \oplus y \geq_T (g \oplus y)'$$

## ④ Determinacy

Given $A \subseteq \omega^\omega \times \omega^\omega$

**Game $G(A)$:**

$$\begin{array}{c|cccc} \mathrm{I} & x_0^{\in\omega} & & x_1 & \cdots \\ \hline \mathrm{II} & & y_0^{\in\omega} & & y_1 \quad \cdots \end{array} \qquad \begin{array}{l} x = x_0 x_1 x_2 \ldots \\ y = y_0 y_1 y_2 \ldots \end{array} \qquad \begin{array}{l} \mathrm{I} \text{ wins if } (x,y) \in A \\ \mathrm{II} \text{ wins otherwise} \end{array}$$

Notation $\sigma$ a strategy for player I and $y \in \omega^\omega$

**$\sigma * y$** denotes the sequence $x = x_0 x_1 x_2 \ldots$

$$\begin{array}{c|ccccc} \mathrm{I} & x_0 = \sigma(\langle\rangle) & & x_1 = \sigma(\langle y_0 \rangle) & & x_2 = \sigma(\langle y_0, y_1 \rangle) \ \ldots \\ \hline \mathrm{II} & & y_0 & & y_1 & \ldots \end{array}$$

Key point 1 $y \mapsto \sigma * y$ is a continuous function

Key point 3 A strategy $\sigma$ can be thought of as an elt of $\omega^\omega$

Key point 2 By determinacy, to get a strategy for player II, it is enough to show how to defeat any fixed strategy for I

## (5) Posner–Robinson $\Rightarrow$ Solecki

(weak) Solecki dichotomy: For all Borel $f: 2^\omega \to 2^\omega$, either $f$ is $\sigma$-continuous or $J \leq_w f$

**Proof strategy** Given $f: 2^\omega \to 2^\omega$ Borel
Define a game $G(f)$ & show that

player I wins $\Rightarrow$ $f$ $\sigma$-continuous
player II wins $\Rightarrow$ $J \leq_w f$

Then use Borel determinacy to finish

When player II wins: $G(f)$ defined to make this easy

When player I wins: Prove the contrapositive
$f$ is not $\sigma$-cont $\Rightarrow$ player I doesn't win

Use Posner-Robinson thm + computability theory characterization of $\sigma$-cont.

**Proof strategy** Define a game $G(f)$ & show that

player I wins $\Rightarrow$ $f$ $\sigma$-continuous

player II wins $\Rightarrow$ $J \le_w f$

**Def** Given $f: 2^\omega \to 2^\omega$ define $G(f)$

| | | | | |
|---|---|---|---|---|
| I | $x_0$ | $x_1$ | $x_2$ | $\cdots$ |
| II $e$ | $y_0, z_0$ | $y_1, z_1$ | $y_2, z_2$ | $\cdots$ |

$x = x_0 x_1 x_2 \ldots$

$y = y_0 y_1 y_2 \ldots \quad z = z_0 z_1 z_2 \ldots$

player II wins if $\Phi_e(f(y), z) = x'$

**Intuitively**
1. Player II claims $J \le_w f$
2. Player I challenges with an input $x$ to $J$
3. Player II responds with an input $y$ to $f$ s.t. $f(y)$ can be used to find $J(x) = x'$
4. $e$ & $z$ describe the continuous function used for $f(y) \rightsquigarrow x'$

## 5.1 When player II wins

**Def** Given $f: 2^\omega \to 2^\omega$ define $G(f)$

| | | | | |
|---|---|---|---|---|
| I | $x_0$ | $x_1$ | $x_2$ | $\cdots$ |
| II | $e$ | $y_0, z_0$ | $y_1, z_1$ | $y_2, z_2$ $\cdots$ |

$x = x_0 x_1 x_2 \ldots$

$y = y_0 y_1 y_2 \ldots \quad z = z_0 z_1 z_2 \ldots$

player II wins if $\Phi_e(f(y), z) = x'$

**Lemma** II has a winning strategy $\Rightarrow J \leq_w f$

pf Suppose II wins via $\tau$

Need to find partial continuous functions $\varphi, \psi$ s.t. for all $x$, $x' = \psi(f(\varphi(x)), x)$

$$\varphi: \ x \longmapsto x * \tau = (e, y, z) \longmapsto y$$

$$\psi: \ f(y), x \longmapsto f(y), \ x * \tau = (e, y, z) \longmapsto \Phi_e(f(y), z)$$

$\tau$ wins $\Rightarrow \Phi_e(f(y), z) = x'$

## 5.2 When player I wins

<u>Def</u> Given $f: 2^\omega \to 2^\omega$ define $G(f)$

| I | $x_0$ | | $x_1$ | | $x_2$ | | $\cdots$ |
|---|---|---|---|---|---|---|---|
| II | $e$ | $y_0, z_0$ | | $y_1, z_1$ | | $y_2, z_2$ | $\cdots$ |

$x = x_0 x_1 x_2 \dots$

$y = y_0 y_1 y_2 \dots \quad z = z_0 z_1 z_2 \dots$

player II wins if $\Phi_e(f(y), z) = x'$

<u>Lemma</u> I has a winning strategy $\Rightarrow$ $f$ is $\sigma$-continuous

We will use:

<u>Lemma</u> $f$ is $\sigma$-continuous $\Leftrightarrow$ for some oracle $a$,
$\forall x,\ f(x) \le_T x \oplus a$

$\sigma$-continuity $\approx$ non-uniform computability

**Lemma** I has a winning strategy $\Rightarrow$ $f$ is $\sigma$-continuous

pf Contrapositive: $f$ not $\sigma$-cont. $\Rightarrow$ player I cannot win

Sps $\gamma$ is a strategy for I. We will show how to defeat $\gamma$.

$f$ not $\sigma$-cont. $\Rightarrow$ $\exists y \; f(y) \not\leq_T y \oplus \gamma$

$f(y) \not\leq_T y \oplus \gamma \Rightarrow \exists g \; f(y) \oplus g \oplus y \oplus \gamma \geq_T (g \oplus y \oplus \gamma)'$

$\hookrightarrow$ relativized Posner-Robinson

$f(y)$ looks like $0'$ rel to $g \oplus (y \oplus \gamma)$

Play as II against $\gamma$:

| | |
|---|---|
| I | $\gamma * (e, y, g \oplus y \oplus \gamma) = x$ |
| II | $e$ $\quad y, g \oplus y \oplus \gamma$ |

Key point: $x \leq_T g \oplus y \oplus \gamma \Rightarrow f(y) \oplus (g \oplus y \oplus \gamma) \geq_T (g \oplus y \oplus \gamma)' \geq_T x'$

How can we pick $e$? Use the recursion thm

## 5.3 Recap

(weak) Solecki dichotomy: For all Borel $f: 2^\omega \to 2^\omega$, either $f$ is $\sigma$-continuous or $J \leq_w f$

Review of the proof:

1. Assume $f$ not $\sigma$-cont.
2. For any strategy $\gamma$ for player I in $G(f)$, can use Posner-Robinson thm to defeat $\gamma$
3. I.e. Posner-Robinson thm shows there is no function witnessing $J \not\leq_w f$
4. Determinacy converts this into a strategy for player II in $G(f)$ i.e. a function witnessing $J \leq_w f$

Determinacy allows us to convert a problem about functions on $2^\omega$ into a problem about elements of $2^\omega$

## 6 Generalizations

**Def** $f: 2^\omega \to 2^\omega$ is $\sigma$-Baire class $\alpha$ if there is a ctbl partition $\langle A_n \rangle_n$ of $2^\omega$ s.t. $\forall n$, $f|A_n$ is Baire class $\alpha$

**Def** $J_\alpha: 2^\omega \to 2^\omega$ denotes $J_\alpha(x) = x^{(\alpha)}$

**Thm** (Marks-Montalbán) For all $f: 2^\omega \to 2^\omega$ Borel and $\alpha < \omega_1$, either $f$ is $\sigma$-Baire class $1+\alpha$ or $J_{\alpha+2} \leq_{sw} f$ I'm sorry :

**Thm** (Shore-Slaman) For all $x$ and $\alpha < \omega_1^{CK}$, either $x \leq_T O^{(\alpha)}$ or there is some $g$ s.t. $x \oplus g \geq_T g^{(\alpha+1)}$

Shore-Slaman + Borel det. $\rightsquigarrow$ weak Marks-Montalbán

$\hookrightarrow$ The proof is almost verbatim from Posner-Robinson/Solecki case

## ⑦ Ordering the Borel functions

Marks–Montalbán ≈ Borel functions are prewellordered by a certain reducibility notion

**Def** $f \le^*_{sW} g$ if $\exists$ ctbl partition $\langle A_n \rangle_{n\in\omega}$ of $2^\omega$ s.t. for all $n$, $f|_{A_n} \le_{sW} g$

non-uniform Weihrauch reducibility

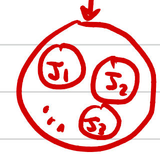
Marks–Montalbán $\Rightarrow$ Every $f: 2^\omega \to 2^\omega$ Borel is $\equiv^*_{sW}$ some $J_\alpha$ or 1 other type of function at limits

$$\text{constant} <^*_{sW} \mathrm{Id} <^*_{sW} J_1 <^*_{sW} J_2 <^*_{sW} \dots < \bigcirc <^*_{sW} J_\omega <^*_{sW} \dots$$

constant ≡ ctbl range

Id ≡ σ-cont.

$J_1$ ≡ σ-Baire class 1




**Question** Under AD, does $\le^*_{sW}$ prewellorder all functions?

↪ the game perspective may help here

Marks-Montalbán $\le_{sW}^*$ prewellorders Borel functions

What about $\le_{sW}$ itself?

<u>Comment</u> It cannot be a prewellorder

Id and $1_A$ are $\le_{sW}$-incomparable

↳ universal $\underset{\sim}{\Pi}^0_1$ set

<u>Question</u> (Carroy) Is $\le_{sW}$ a well-quasi-order on Borel fns?

Some reducibility notions

| | | Borel | all (under AD) |
|---|---|---|---|
| $\le_{sW}$ | Weihrauch reducibility | ? | ? |
| | Carroy: Yes for cts functions w/ ctbl range | | |
| $\le_{sW}^*$ | non-uniform Weihrauch | ✓ | ? |
| | Marks-Montalbán | | |
| $\le_{sW}^{\wedge}$ | parallelized Weihrauch | ✓ | ✓ |
| | Day-Downey-Westrick/Kihara/Steel/Becker | | |

## ⑧ Questions

① Can these proofs be extended to prove the full Solecki/Marks-Montalbán thms? ↳ E.g with $\leq_{sw}$ instead of $\leq_w$

② It is not hard to modify $G(f)$ to a game $G(f,g)$ which characterizes $\leq_w$
i.e. II has a winning strategy in $G(f,g) \iff f \leq_w g$

Can this be done for $\leq_{sw}$ as well?
↳ this might help answer ①

③ Is the proof of the Solecki dichotomy using the Posner-Robinson thm part of a more general pattern?