Question Is the Lebesgue ultrafilter on the Turing degrees Rudin-Keisler below Martin measure?

Goal of this talk: Explain what this question means and why it's interesting.

Martin's Conjecture and Ultrafilters on the Turing Degrees

① Martin's Conjecture

What operations on the Turing degrees can you think of?

$x \longmapsto c$ constant functions

$x \longmapsto x$ identity

$x \longmapsto x'$ jump

$x \longmapsto x''$ double jump

$\vdots$

$x \longmapsto x^{(\omega)}$ ω-jump

$x \longmapsto \mathcal{O}^x$ hyperjump

$\vdots$

<u>Motivating question:</u> Is this everything? Are we missing any fundamental operations on the Turing degrees?

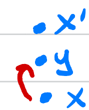
Turing degrees

<u>Naive conjecture:</u> For all functions $f : \mathcal{D}_T \to \mathcal{D}_T$, either $f(x) \leq_T x$ for all x or $f(x) \geq_T x'$ for all x

↳ Totally false!

Naive conjecture: For all functions $f: \mathcal{D}_T \to \mathcal{D}_T$, either $f(x) \leq_T x$ for all x or $f(x) \geq_T x'$ for all x

Counterexample 1: By the Kleene-Post theorem, for each x we can pick some y such that $x <_T y <_T x'$. Mapping each x to such a y gives a function strictly in-between the identity & the jump



↪ Requires AC

Counterexample 2: Let a be some fixed Turing degree. Define

$$f(x) = \begin{cases} x & \text{if } x \not\geq_T a \\ x' & \text{if } x \geq_T a \end{cases}$$

Sometimes equal to the identity, sometimes equal to the jump

↪ Equal to the jump once you get above a.

Martin's Conjecture: ① Replace AC with AD ← Axiom of Determinacy

② Look at behavior of functions "in the limit"

(1.1) What Does "In the Limit" Mean?

<u>Def</u> A **cone** of Turing degrees is a set of the form

$\{x \mid x \geq_T y\}$ for some fixed y

$\uparrow$ Cone(y)

<u>Def</u> If $f, g : \mathcal{D}_T \to \mathcal{D}_T$:

$f \leq_M g$ means $f(x) \leq_T g(x)$ on a cone

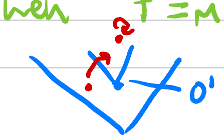
$f \equiv_M g$ means $f(x) \equiv_T g(x)$ on a cone

$\Leftrightarrow$ $f \leq_M g$ and $g \leq_M f$

"Martin order" and "Martin equivalence"

"$f = g$ in the limit" = "$f \equiv_M g$"

<u>Example</u> Let $f(x) = x \oplus 0'$. Then $f \equiv_M \mathrm{id}$ because they agree on Cone(0')



1.2 The Axiom of Determinacy: Why?

- AD says that in certain types of games, one player always has a winning strategy
- Contradicts AC
- Equiconsistent with a certain large cardinal principle

Implies many nice regularity properties for sets of reals

Every set of reals is Lebesgue measurable, has the property of Baire, etc.

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ZFC

ZF

ZF + AD

(1.2) The Axiom of Determinacy: Why?

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Why use it?

(1) Consistency result → Like using CH, V=L, MA, etc.

(2) Borel determinacy provable in ZF
Determinacy for $L(\mathbb{R})$ provable from large cardinals

(3) It's useful!

Disclaimer: We will sometimes use $AD_{\mathbb{R}}$, a strengthening of AD

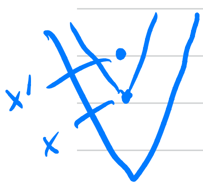
(1.3) The Axiom of Determinacy: How?

Thm (Martin's cone theorem) AD implies that every set of Turing degrees either contains a cone or is disjoint from a cone.

Not disjoint from a cone = $\forall x\ \exists y \geq_T x\ \ y \in A$

↳ "cofinal"

Thm restated: $A \subseteq \mathcal{D}_T$ cofinal $\Rightarrow$ A contains a cone



Example (Jump inversion via Nuclear flyswatter) Every large enough degree is the jump of something

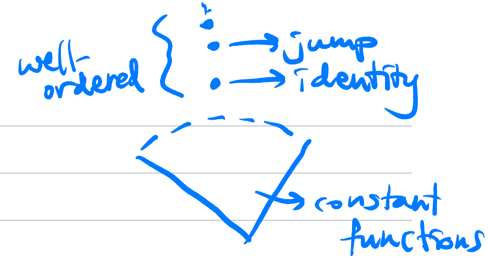
pf: Want to show

$$A = \{x \mid \exists y\ \ y' = x\}$$

contains a cone. By determinacy, enough to show it's cofinal. Let $x \in \mathcal{D}_T$. Then x' is above x and in A. $\square$

↳ Kind of silly. Friedberg jump inversion shows $A = \mathrm{Cone}(0')$

1.4 Statement of Martin's Conjecture



Conjecture (Martin) Assume ZF + AD

① Every function $f : \mathcal{D}_T \to \mathcal{D}_T$ is either

- Martin equivalent to a constant function
  $f(x)$ constant on a cone
- Martin above the identity
  $f(x) \geq_T x$ on a cone

② Functions which are above the identity are prewellordered by $\leq_M$ and the successor in this order is given by the jump

successor of f is $x \mapsto f(x)'$

Disclaimer: Not the usual statement. Equivalent under $AD_{\mathbb{R}}$

② Some Results on Part 1 of Martin's Conjecture

Thm (Slaman–Steel) Part 1 of Martin's conjecture holds for all regressive functions on the Turing degrees

Thm (L.–Siskind) Part 1 of Martin's conjecture holds for all measure preserving functions on the Turing degrees

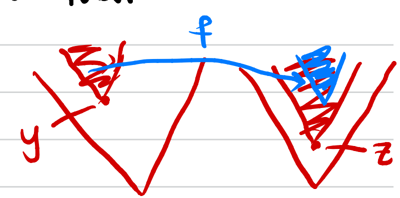
under $AD_{\mathbb{R}}$

Def A function $f : \mathcal{D}_T \to \mathcal{D}_T$ is regressive if $f(x) \leq_T x$ for all x on a cone

"f is computable"

Def A function $f : \mathcal{D}_T \to \mathcal{D}_T$ is measure preserving if for all z, there is some y such that

$$x \geq_T y \Rightarrow f(x) \geq_T z$$



"f eventually gets above z"
"f is going to infinity in the limit"

③ Martin measure

<u>Def</u> Martin measure, denoted $\mathcal{U}_M$, is the collection of subsets of $\mathcal{D}_T$ such that

$$A \in \mathcal{U}_M \iff A \text{ contains a cone}$$

"A is big if A contains a cone"

<u>Thm</u> (Martin's cone thm, restated) $\mathcal{U}_M$ is an ultrafilter

Actually, a countably complete ultrafilter

$\text{Cone}(x) \cap \text{Cone}(y) = \text{Cone}(x \oplus y)$

$\bigcap \text{Cone}(x_i) \supseteq \text{Cone}(\bigoplus x_i)$

"$f \leq_M g$" $=$ $f(x) \leq_T g(x)$ $\mathcal{U}_M$-almost everywhere

"$f \equiv_M g$" $=$ $f(x) = g(x)$ $\mathcal{U}_M$-almost everywhere

④ Ultrapower by the Martin Measure

If we have an ultrafilter, we can take ultrapowers

Ultrapower of $\mathcal{D}_T$ by $\mathcal{U}_M$:

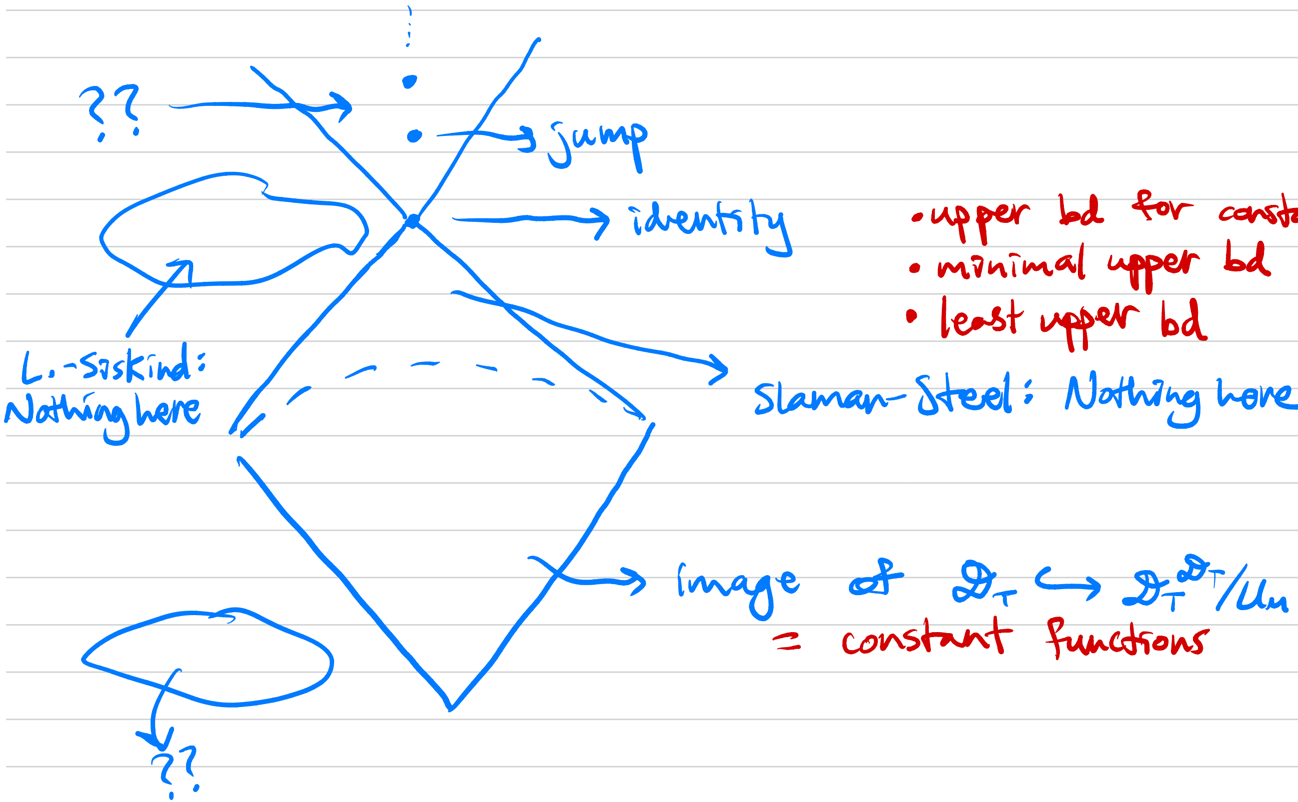
① $f : \mathcal{D}_T \to \mathcal{D}_T$ = representative of an element of $\mathcal{D}_T^{\mathcal{D}_T}/\mathcal{U}_M$

② $f \equiv_M g$ = $[f]_{\mathcal{U}_M} = [g]_{\mathcal{U}_M}$

③ $f \leq_M g$ = $[f]_{\mathcal{U}_M} \leq_T [g]_{\mathcal{U}_M}$

④ f measure preserving = $[f]_{\mathcal{U}_M}$ is above the image of $\mathcal{D}_T \hookrightarrow \mathcal{D}_T^{\mathcal{D}_T}/\mathcal{U}_M$

A picture of $\mathcal{D}_T^{\mathcal{D}_T}/\mathcal{U}_M$:



- upper bd for constant functions
- minimal upper bd
- least upper bd

⑤ The Pushforward of an Ultrafilter

Def If $\mathcal{U}$ is an ultrafilter on a set X and $f: X \to Y$ then the pushforward of $\mathcal{U}$ along f, written $f_*(\mathcal{U})$, is the ultrafilter on Y defined by

$$\mathcal{U} \rightsquigarrow f_*(\mathcal{U}) \qquad X \xrightarrow{f} Y$$

$$A \in f_*(\mathcal{U}) \iff f^{-1}(A) \in \mathcal{U}$$

Example If $f: X \to Y$ is constant on a set in $\mathcal{U}$ then $f_*(\mathcal{U})$ is principal

Example The pushforward of Martin measure along $x \longmapsto \omega_1^x$ gives a countably complete ultrafilter on ω_1

Hence $AD \Rightarrow \omega_1$ is a measurable cardinal!

Note: $f: X \to Y$ induces an embedding $M^Y/f_*(\mathcal{U}) \hookrightarrow M^X/\mathcal{U}$

$$[g]_{f_*(\mathcal{U})} \longmapsto [g \circ f]_{\mathcal{U}}$$

Def A function $f : X \to X$ is measure preserving for an ultrafilter $\mathcal{U}$ on X if

$$f_*(\mathcal{U}) = \mathcal{U} \iff f(A) \in \mathcal{U} \text{ for all } A \in \mathcal{U}$$

Fact $f : \mathcal{D}_T \to \mathcal{D}_T$ is measure preserving if and only if it is measure preserving for Martin measure

⑥ The Rudin-Keisler Order

Def If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on a set X

$\mathcal{U} \leq_{RK} \mathcal{V}$ means there is $f : X \to X$ s.t. $f_*(\mathcal{V}) = \mathcal{U}$

seems backwards. But remember $f_*(\mathcal{V}) = \mathcal{U} \Rightarrow M^X/\mathcal{U} \hookrightarrow M^X/\mathcal{V}$

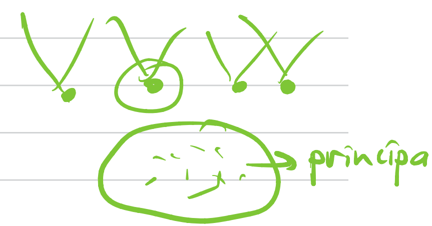
$\mathcal{U} \equiv_{RK} \mathcal{V}$ means $\mathcal{U} \leq_{RK} \mathcal{V}$ and $\mathcal{V} \leq_{RK} \mathcal{U}$

↳ not the usual definition

$\leq_{RK}$ is a quasi order on the set of ultrafilters on X

Example If $\mathcal{U}$ is a principal ultrafilter, it is $\leq_{RK}$ all other ultrafilters

Example $\leq_{RK}$-minimal nonprincipal ultrafilters



- On ω: the Ramsey ultrafilters
- On κ: the normal ultrafilters

Thm (Part 1 of Martin's conjecture for measure preserving functions, restated)

Part 1 of Martin's conjecture $\iff$ If $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathcal{D}_T$ s.t. $\mathcal{U} \leq_{RK} \mathcal{U}_M$ then $\mathcal{U} = \mathcal{U}_M$

$\mathcal{U}_M$ is $\leq_{RK}$-minimal among nonprincipal ultrafilters on $\mathcal{D}_T$ AND nothing else is $\equiv_{RK}$ to $\mathcal{U}_M$

proof 3 cases

1. $f_*(\mathcal{U}_M)$ principal $\iff$ f constant on a cone
2. $f_*(\mathcal{U}_M) = \mathcal{U}_M$ $\iff$ f measure preserving $\iff$ $f \geq_M \mathrm{id}$
3. $f_*(\mathcal{U}_M) \neq \mathcal{U}_M$ & nonprincipal $\Rightarrow$ $f_*(\mathcal{U}_M) \leq_{RK} \mathcal{U}_M$

This suggests: Try to prove part 1 of Martin's conjecture by looking at $\leq_{RK}$ on the Turing degrees

- Use ideas from set theory
- Just prove $\mathcal{U}_M$ is $\leq_{RK}$ minimal or just prove nothing is $\equiv_{RK}$ to it
- Look at specific ultrafilters on $\mathcal{D}_T$ and try to show they are not below $\mathcal{U}_M$

⑦ The Lebesgue Ultrafilter

Def $x, y \in 2^\omega$ are called tail-equivalent if they agree on all but finitely many positions

→ Note: if x & y tail equivalent then $x \equiv_T y$

Example

$x = 0\,0\,0\,0\,0\,0\,0\,0\,0 \ldots\ldots$
$y = 1\,0\,1\,0\,0\,0\,0\,0 \ldots\ldots$

Thm (Kolmogorov 0-1 Law) If $A \subseteq 2^\omega$ is Lebesgue measurable & closed under tail equivalence then either A has measure 0 or measure 1

Thm AD $\Rightarrow$ Every subset of 2^ω is Lebesgue measurable

Cor Lebesgue measure induces an ultrafilter on $\mathcal{D}_T$!

$A \in \mathcal{U}_L \Leftrightarrow \{x \mid \deg_T(x) \in A\}$ has measure 1

Question: Is $\mathcal{U}_L \leq_{RK} \mathcal{U}_M$?

What we can show: $\mathcal{U}_M \nleq_{RK} \mathcal{U}_L$ $(\Rightarrow \mathcal{U}_M \not\equiv_{RK} \mathcal{U}_L)$

Thm $u_M \not\leq_{RK} u_L$

Lemma (ZF + AD) Any $f : 2^\omega \to \omega_1$ is constant on a set of positive measure

proof Suppose not.

$\Rightarrow \forall \alpha \;\; \mu(f^{-1}(\alpha)) = 0$

$\Rightarrow \forall \alpha \;\; \mu(\{x \mid f(x) \leq \alpha\}) = \sum_{\beta \leq \alpha} \mu(f^{-1}(\beta)) = 0$

$B = \{(x,y) \mid f(x) \leq f(y)\}$

By Fubini,

$$\mu(B) = \int \overset{=0}{\mu(\{x \mid (x,y) \in B\})}\, dy = \int \overset{=1}{\mu(\{y \mid (x,y) \in B\})}\, dx$$

$\{x \mid f(x) < f(y)\}$ measure 0

$\{y \mid f(x) < f(y)\}$ measure 1

Thm $\mathcal{U}_M \not\leq_{RK} \mathcal{U}_L$

Lemma (ZF + AD) Any $f : 2^\omega \to \omega_1$ is constant on a set of positive measure

proof of thm Suppose $f_*(\mathcal{U}_L) = \mathcal{U}_M$

Consider
$$\begin{aligned} 2^\omega &\longrightarrow \omega_1 \\ x &\longmapsto \omega_1^{f(\deg_T(x))} \end{aligned}$$

Constant on a set of positive measure
$\Rightarrow$ Constant on a set of measure 1
$\Rightarrow$ $\exists A \in \mathcal{U}_L$ s.t. $\forall x \in A$ $\omega_1^{f(x)} = \alpha$
$\Rightarrow$ $f(A)$ disjoint from a cone
$\Rightarrow$ $f_*(\mathcal{U}_L) \neq \mathcal{U}_M$

⑧ Questions

Question: Is $\mathcal{U}_L$ strictly below $\mathcal{U}_M$?

Thm (Marks) $\mathcal{U}_L <_{RK} \mathcal{U}_M \iff \exists f : 2^\omega \to 2^\omega$ Turing invariant $f(x)$ is x-random for all x

Assuming $AD_{\mathbb{R}}$

Question: Is $\mathcal{U}_M$ $\leq_{RK}$-maximal among ultrafilters on $\mathcal{D}_T$?