Lossless expansion and Measure Hyperfiniteness

Joint work with
Jan Grebík

① Countable Borel equivalence relations (CBER)

X, Y standard Borel spaces

E, F equivalence relations on X, Y

E is a CBER $\quad E \subseteq X \times X$ is Borel

All equivalence classes countable

Borel reducible

$E \leq_B F \quad \exists f : X \to Y$ Borel s.t. $xEy \Rightarrow f(x) F f(y)$

Examples

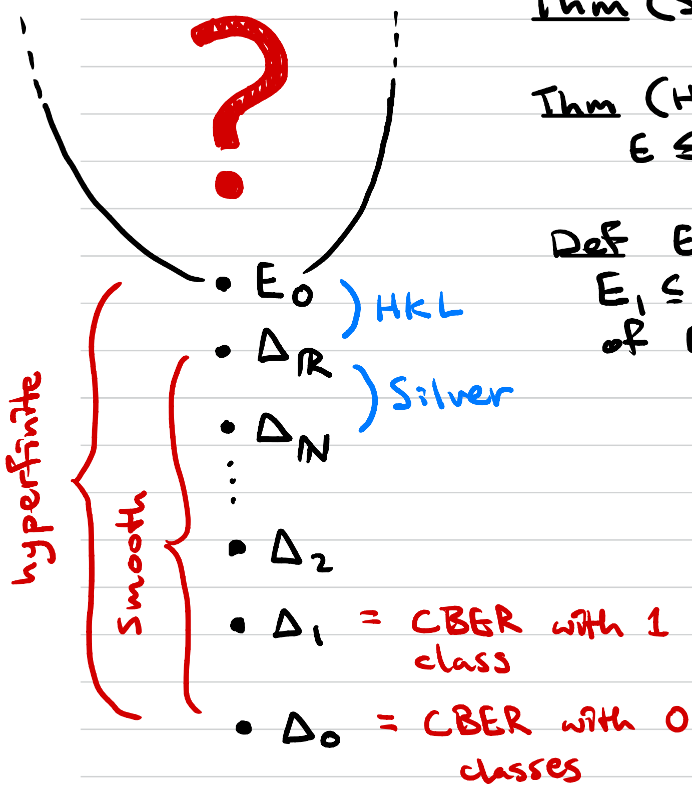
① Δ_X = equality on $X \quad x \sim y \Leftrightarrow x = y$

② $\Delta_{\mathbb{N}} \equiv_B$ any CBER with countably many classes

③ E_0 = eventual equality on $2^{\mathbb{N}}$

$f \sim g \Leftrightarrow \exists N \, \forall n \geq N \; f(n) = g(n)$

(1.1) Picture of CBERs



Thm (Silver) Either $E \leq_B \Delta_{\mathbb{N}}$ or $\Delta_{\mathbb{R}} \leq_B E$

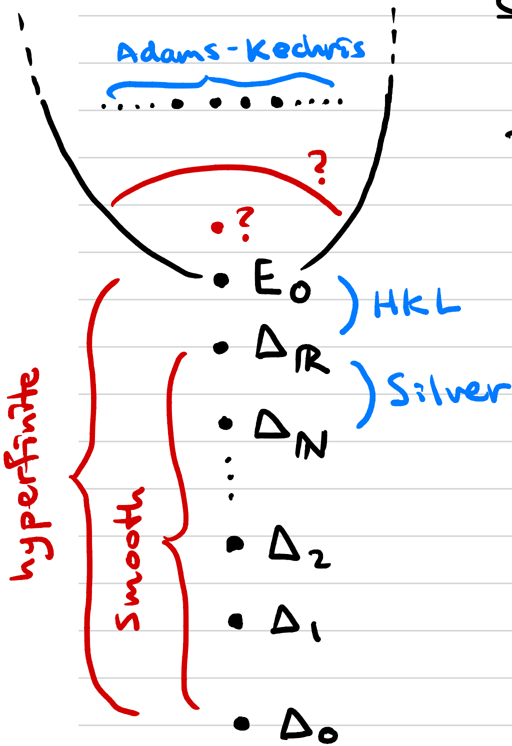
Thm (Harrington-Kechris-Louveau) Either $E \leq_B \Delta_{\mathbb{R}}$ or $E_0 \leq_B E$

Def E is **hyperfinite** if there are $E_1 \subseteq E_2 \subseteq \ldots$ s.t. all equivalence classes of E_n are finite & $E = \bigcup_n E_n$

Thm (Dougherty-Jackson-Kechris) E hyperfinite $\Leftrightarrow$ $E \leq_B E_0$

Question What is the structure of non-hyperfinite CBERs?

(1.1) Picture of CBERs



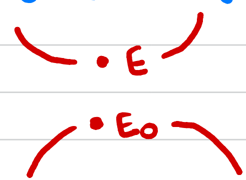
Question What is the structure of non-hyperfinite CBERs?

Thm (Adams-Kechris) There is an uncountable antichain of CBERs → Actually much more

Questions

1. More dichotomy theorems?

   $E >_B E_0$ s.t. $\forall F \,(F \le_B E_0$ or $E \le_B F)$



2. Successor of E_0?

   $E >_B E_0$ s.t. $F <_B E \Rightarrow F \le_B E_0$?

② Measure reducibility

Comment All known proofs of $E \not\leq_B F$ use measure theory
for E, F non-smooth

Idea Study $\leq_B$ up to measure zero

μ Borel probability measure on X

$E \leq_\mu F$ $\exists A \subseteq X$ s.t. $\mu(A) = 1$ and $E|_A \leq_B F$

$E \leq_M F$ For all μ, $E \leq_\mu F$

E is μ-hyperfinite $\exists A \subseteq X$ s.t. $\mu(A)=1$ and $E|_A$ hyperfinite

E is measure hyperfinite For all μ, E is μ-hyperfinite

Comment E measure hyperfinite $\Leftrightarrow$ $E \leq_M E_0$

(2.1) Conley and Miller's results

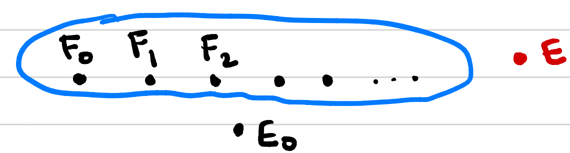
Question More dichotomy theorems?

$E >_{\cancel{B}\,M} E_0$ s.t. $\forall F\ (F \leq_{\cancel{B}\,M} E_0$ or $E \leq_{\cancel{B}\,M} F)$

Thm (Conley-Miller) No countable base for non-measure-hyperfinite CBERs under $\leq_M$

I.e. for any $(F_n)_{n\in\mathbb{N}}$ non-measure-hyperfinite, $\exists E$ s.t. for all n, $F_n \not\leq_M E$

$\Rightarrow$ No dichotomy thm



The point Any further dichotomy cannot use measure theory

Question Successor of E_0?

$E >_{\cancel{B}\,M} E_0$ s.t. $F <_{\cancel{B}\,M} E \Rightarrow F \leq_{\cancel{B}\,M} E_0$?

This talk Probably yes.

2.2 Successors of E_0

Question Successor of E_0?

$E >_M E_0$ s.t. $F <_M E \Rightarrow F \leq_M E_0$?

Thm (Conley–Miller) Suppose λ is a measure s.t.

① E is not λ-hyperfinite

② For all $\mu \perp \lambda$, E is μ-hyperfinite

Then E is a successor of E_0 for $\leq_M$

$\exists A$ $\mu(A) = 1$, $\lambda(A) = 0$

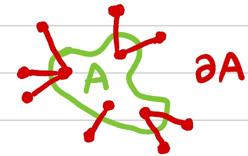
This talk (me + Jan Grebík):

① A combinatorial condition that implies Conley & Miller's condition

lossless expansion

② Two plausible candidates for this combinatorial condition

③ Lossless expansion



$G = (V, E)$ finite d-regular graph

A $A \subseteq V$

∂A $\{e \mid$ one endpoint of e in A, one in $V - A\}$

Def (Edge expansion) $h(G) = \min_{|A| \leq |V|/2} |\partial A| / |A|$

$h(G)$ large $\Rightarrow$ hard to trap a random walk in a set A

Comment Random d-regular graph has high expansion
$h(G) \approx d/2$

Comment average degree of $A = d - |\partial A|/|A|$

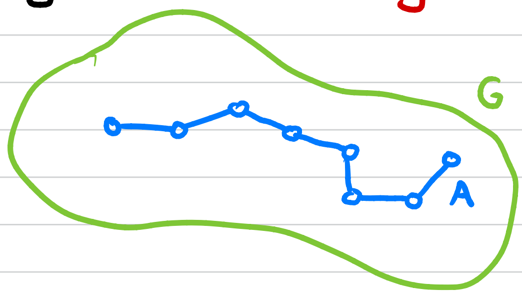
(Informal) **Def** G is a lossless expander if very small subsets of G have almost optimal expansion $\rightarrow$ average degree $\leq 2 + \varepsilon$

$|\partial A|/|A| \geq d - 2 - \varepsilon$

(Informal) **Def** G is a lossless expander if very small subsets of G have almost optimal expansion → average degree $\leq 2+\varepsilon$

Question Why $\leq 2+\varepsilon$? Why not $\leq 1+\varepsilon$?

Answer



average degree of A: $2-\varepsilon$

(Non-standard) **Def** A family of d-regular graphs $G_0, G_1, \ldots$ is a lossless expanding family if for all $\varepsilon > 0$ there is $\delta > 0$ and N s.t.

$$n \geq N, \quad A \subseteq V(G_n), \quad |A| \leq \delta |V(G_n)| \Rightarrow \text{average deg. of } A \leq 2+\varepsilon$$

(Non-standard)

<u>Def</u> A family of d-regular graphs $G_0, G_1, \ldots$ is a lossless expanding family if for all $\varepsilon > 0$ there is $\delta > 0$ and N s.t.

$$n \geq N,\ A \subseteq V(G_n),\ |A| \leq \delta |V(G_n)| \Rightarrow \text{average deg. of } A \leq 2 + \varepsilon$$

<u>Example</u> G_n = random d-regular graph on n vertices

w.h.p. $G_0, G_1, \ldots$ is a lossless expander

Recently: Lossless expanders used to construct good quantum error-correcting codes

④ Lossless expansion in Borel graphs

G d-regular Borel graph on X

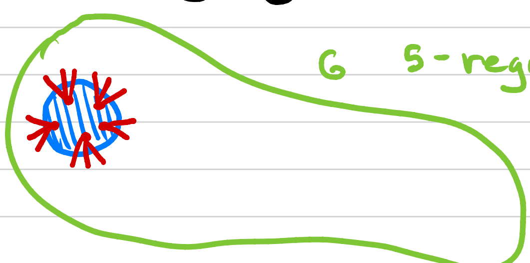
λ Borel probability measure on X

$$\left.\begin{aligned} \deg_A(x) &= |\{y \in A \mid (x,y) \in E(G)\}| \\ \operatorname{avg}\deg_\lambda(A) &= \int_A \deg_A(x)\, d\lambda \end{aligned}\right\} \quad A \subseteq X \text{ Borel}$$

Def G is a λ-lossless expander if for all $\varepsilon > 0$ there is $\delta > 0$ such that $\lambda(A) \leq \delta \Rightarrow \operatorname{avg}\deg_\lambda(A) \leq 2 + \varepsilon$

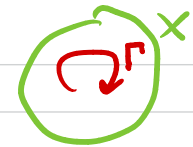
Comment $N(A) = A \cup \{y \mid \exists x \in A \ (x,y) \in E(G)\}$

$\operatorname{avg}\deg_\lambda(A) \leq 2 + \varepsilon \Rightarrow \lambda(N(A)) \geq (d-1-\varepsilon)\lambda(A)$



G 5-regular $\rightsquigarrow$ most vertices in A have ≥ 3 neighbors outside A

4.1 Lossless expansion and successors of E_0



X compact Polish space with fixed metric

Γ finitely-generated non-amenable group acting freely on X

G Schreier graph of $\Gamma \curvearrowright X$

λ Γ-invariant probability measure on X s.t. $\mathrm{supp}(\lambda) = X$

E orbit equivalence relation of $\Gamma \curvearrowright X$

Thm (Grebík-L.) If Γ acts by isometries and G is a λ-lossless expander then E is a successor of E_0 for $\leq_M$

Idea Γ non-amenable, acts freely $\Rightarrow$ not λ-hyperfinite

$\mu \perp \lambda$, λ-lossless expander $\Rightarrow$ μ-hyperfinite

To finish, apply Conley & Miller's thm

4.2 Proving hyperfiniteness

Two useful ideas when proving μ-hyperfiniteness

① **Def** An undirected Borel graph G is orientable if its edges can be directed such that each vertex has out degree at most 1

Thm (Dougherty-Jackson-Kechris) If G is Borel orientable then the associated equivalence relation is hyperfinite

② To show E is μ-hyperfinite, it is enough to show that for all $\varepsilon > 0$ there is A s.t. $\mu(A) \geq 1-\varepsilon$ and $E|_A$ is hyperfinite

(Because we can assume μ quasi-invariant)

↳ Essentially Dye-Krieger

4.3 Proof sketch

<u>Thm</u> (Grebík–L.) If Γ acts by isometries and G is a λ-lossless expander then E is a successor of E_0 for $\leq_M$

Fix $\mu \perp \lambda$, $\varepsilon > 0$

Goal: Find $A \subseteq X$ s.t. ① $\mu(A) \geq 1-\varepsilon$
② $G|_A$ Borel orientable

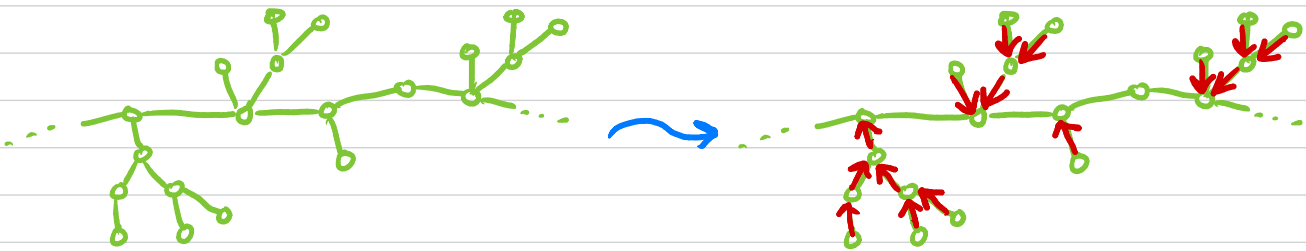
Iterative process: On each step, delete a small number of vertices & orient some edges

To ensure $\mu(A) \geq 1-\varepsilon$: On each step, many more edges oriented than vertices deleted

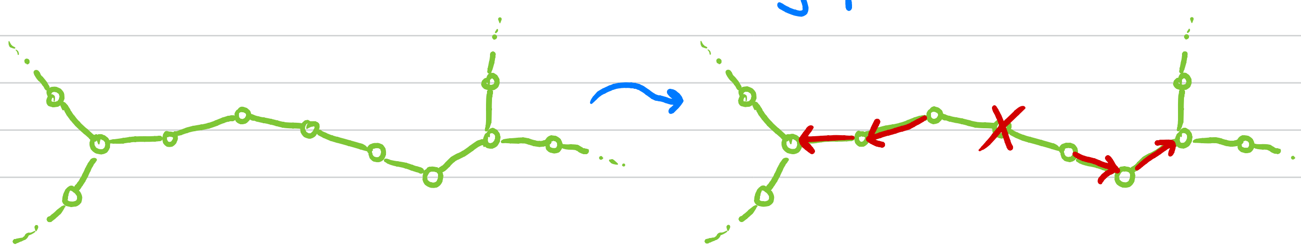
Iterative process: On each step, delete a small number of vertices & orient some edges

Each step:

Phase 1 Iteratively orient degree 1 vertices



Phase 2 Cut & orient long paths



Iterative process: On each step, delete a small number of vertices & orient some edges

Claim 1: This produces an orientation

We only orient an edge away from a vertex when the vertex has degree 1

Claim 2: We never get stuck

There is always either a deg. 1 vertex or a long path

If not, get a set with high average degree and λ-measure 0

all vertices deg. ≥ 2, lots of high deg. vertices

Contradicts lossless expansion

After taking δ-thickening for some small enough δ

⑤ Candidates

Thm (Grebík–L.) If Γ acts by isometries and G is a λ-lossless expander then E is a successor of E_0 for $\leq_M$

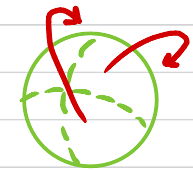
Question Does this ever actually happen?

Two candidates:

1. Random rotations of S^2
2. Limit of sequence of finite graphs

5.1 Random rotations of S^2

Pick two rotations $\gamma_0, \gamma_1 \in SO(3)$



$\Gamma = \langle \gamma_0, \gamma_1 \rangle$
$X = S^2$
$\lambda =$ Lebesgue measure

<u>Fact</u> If we pick two rotations of S^2 at random then with probability 1, they generate a free subgroup of $SO(3)$

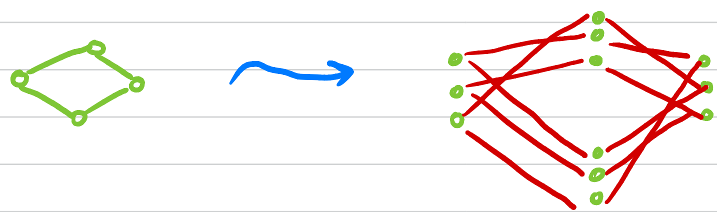
Bourgain–Gamburd: Many examples of 2 rotations which generate expander graphs

but not necessarily lossless expanders

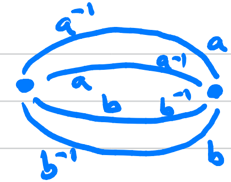
5.2 Limit of finite graphs

Def Given a finite graph G, a **k-lift** $G' \twoheadrightarrow G$ is a graph formed from G by:

1. Replace every vertex u of G by k vertices $u_1, \ldots, u_k$
2. Replace every edge (u,v) of G by a matching of $\{u_1, \ldots, u_k\}$ & $\{v_1, \ldots, v_k\}$



If matchings are chosen randomly, G' is a **random k-lift** of G

Idea ① Start with $G_0 =$ 

Note: $\mathbb{F}_2$ acts on G_0

② Form $\underbrace{G_0 \twoheadleftarrow G_1}_{\substack{\text{random} \\ k_0\text{-lift}}} \underbrace{\twoheadleftarrow G_2}_{\substack{\text{random} \\ k_1\text{-lift}}} \twoheadleftarrow \ldots$ ...

$k_0, k_1, k_2, \ldots$ fast-growing sequence

③ $G = \varprojlim G_n$

Note: $\mathbb{F}_2$ acts on G, freely w/ prob. 1

④ $\lambda =$ natural measure on $G =$ limit of counting measures on G_n

Some evidence G is a λ-lossless expander