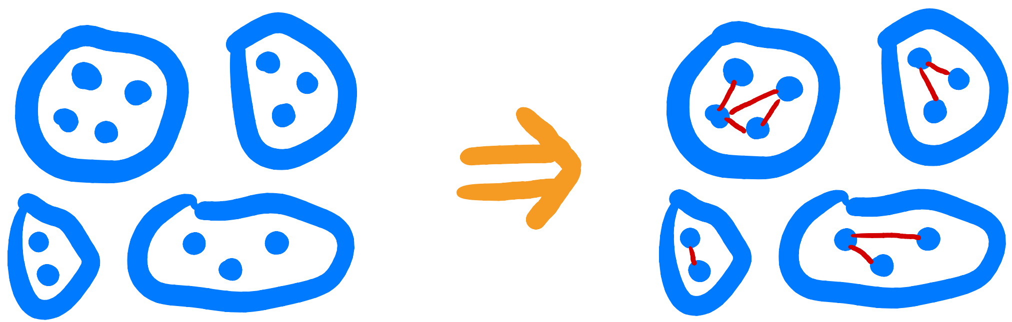
Borel graphable equivalence relations



Joint work w/ Tyler Arant & Alexander Kechris

① Introduction

standard Borel space

Def (Arant) An equivalence relation E on X is Borel graphable if there is a Borel graph $G \subseteq X \times X$ such that

E = connectivity in G

$xEy \Leftrightarrow \exists$ path in G between x & y

Observation E Borel graphable $\Rightarrow$ E is analytic

Question Borel $\subseteq$ Borel graphable $\subseteq$ Analytic

$\subsetneq$? $\qquad \subsetneq$?

Answers Yes to both

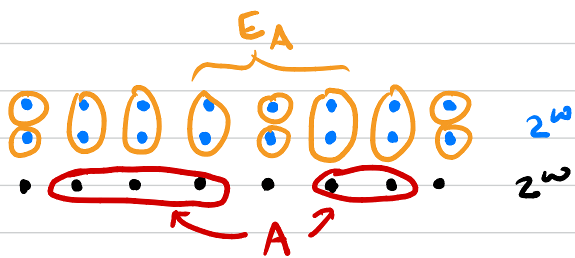
There is a non-Borel E which is Borel graphable

There is an analytic E which is not graphable

Prop Not every analytic equivalence relation is Borel graphable

pf $A \subseteq 2^\omega$ analytic, not Borel

E_A on $2^\omega \times 2$:



$x \in A \Rightarrow (x,0) \; E_A \; (x,1)$

$x \notin A \Rightarrow (x,0) \; \not{E}_A \; (x,1)$

Note: E_A Borel graphable $\Leftrightarrow$ E_A Borel

$\Leftrightarrow$ A Borel

Comment If all equivalence classes of E are ctbl then
E Borel graphable $\Leftrightarrow$ E is Borel

By Lusin-Novikov

universal for Borel reducibility

Prop There is a universal analytic equivalence which is Borel graphable

pf Pick E on 2^ω universal analytic
Define E' on $2^\omega \times 2^\omega$ by

$$(x_0, y_0) \; E' \; (x_1, y_1) \iff x_0 \, E \, x_1$$

2nd coord. is ignored

① E' still universal

② Define G by

$((x_0, y_0), (x_1, y_1)) \in G \iff$ $y_0 \oplus y_1$ computes a witness that $x_0 E x_1$

Given $(x_0, y_0) \, E' \, (x_1, y_1)$, let y_2 compute a witness that $x_0 \, E \, x_1$

Then $(x_0, y_0) - (x_0, y_2) - (x_1, y_1)$ is a path in G

Def (Arant) An equivalence relation E on X is **Borel graphable** if there is a Borel graph $G \subseteq X \times X$ such that
E = connectivity in G

Prop Borel $\subsetneq$ Borel graphable $\subsetneq$ analytic

add a "dummy coordinate"   make all classes ctbl

But these examples are kind of disappointing...

Question More interesting examples of Borel graphability / non-graphability?

Two case studies: (1) Countable admissible ordinals

(2) Polish group actions

② First example: F_{ω_1}

Recall ω_1^{CK} = least ordinal w/ no computable presentation
= least ordinal w/ no Δ^1_1 presentation

for $x \in 2^\omega$

ω_1^x = least ordinal w/ no presentation computable from x
= least ordinal w/ no $\Delta^1_1(x)$ presentation

Fact (Spector) $x \in \Delta^1_1(y) \Rightarrow \omega_1^x \leq \omega_1^y$

$x \leq_H y$ "hyperarithmetic in y"

Def F_{ω_1} is the equivalence relation on 2^ω defined by

$$x \; F_{\omega_1} \; y \iff \omega_1^x = \omega_1^y$$

Facts F_{ω_1} is ...

1. analytic, but not Borel
2. all equivalence classes are Borel
3. ω_1-many equivalence classes
4. not induced by a Polish grp action

counterexample to a strong form of Vaught's conjecture

Marker

(2.1) F_{ω_1} is not Borel graphable

Def $x \, F_{\omega_1} \, y \iff \omega_1^x = \omega_1^y$

Thm (Arant) Assume $V = L$. Then F_{ω_1} is not Borel graphable

Proof idea: (1) Prove in ZFC that F_{ω_1} is not Δ^1_1 graphable

(2) Use a special fact about L to adapt the proof to show that F_{ω_1} is not $\Delta^1_1(x)$ graphable for any $x \in 2^\omega$

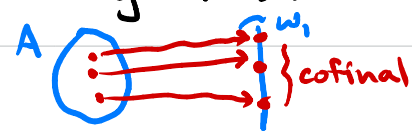
Key tools:

Martin/Friedman

Thm (Effective perfect set thm) If $A \subseteq 2^\omega$ is Σ^1_1 then either A contains a perfect set or every elt of A is Δ^1_1

Thm (Friedman's conjecture) If $A \subseteq 2^\omega$ is Δ^1_1 and unctbl then for every x, there is $y \in A$ s.t. x is $\Delta^1_1(y)$

$\Rightarrow \forall \alpha < \omega_1 \ \exists y \in A \quad \alpha < \omega_1^y$



Thm (Arant) F_{ω_1} is not Δ^1_1-graphable

pf Suppose for contradiction G is a Δ^1_1-graphing of F_{ω_1}

Claim Every elt of connected component of 0 is Δ^1_1

pf By induction

Suppose x is Δ^1_1, need to prove all neighbors are Δ^1_1

$N(x) = \{ y \in 2^\omega \mid (x,y) \in G \}$ is $\Delta^1_1(x) = \Delta^1_1$

Eff. perfect set thm $\Rightarrow$ $N(x) \subseteq \Delta^1_1$ or unctbl

Case 1: $N(x) \subseteq \Delta^1_1$. We are done

Case 2: $N(x)$ unctbl. Friedman's conj. $\Rightarrow$ $\exists y \in N(x)\ \omega_1^x < \omega_1^y$

Contradicts $x \ F_{\omega_1} \ y$

By claim, $[0]_{F_{\omega_1}}$ = connected component of 0 in $G \subseteq \Delta^1_1$

But it is known $[0]_{F_{\omega_1}} \nsubseteq \Delta^1_1$ $\rightarrow\leftarrow$

Goal: F_{ω_1} not Borel graphable

Idea: Modify proof that F_{ω_1} not Δ^1_1-graphable to show that for any $a \in 2^\omega$, F_{ω_1} is not $\Delta^1_1(a)$-graphable

Problem: Friedman's conjecture

<u>Friedman's conjecture</u> If $A \subseteq 2^\omega$ is Δ^1_1 & unctbl then for every $\alpha < \omega_1$, $\exists x \in A$ s.t. $\alpha < \omega_1^x$

<u>Relativised</u> If $A \subseteq 2^\omega$ is $\Delta^1_1(a)$ & unctbl then for every $\alpha < \omega_1$, $\exists x \in A$ $\alpha < \omega_1^{x \oplus a}$ but in general, $\omega_1^x < \omega_1^{x \oplus a}$

Solution: We can assume that for all $x \in A$, $\omega_1^x = \omega_1^a$ + use the following thm

<u>Thm</u> Assume $V = L$. For every $a \in 2^\omega$, there is $b \in 2^\omega$ s.t. a is $\Delta^1_1(b)$ and $\omega_1^x \geq \omega_1^b \Rightarrow b$ is $\Delta^1_1(x)$ $\Rightarrow \omega_1^x = \omega_1^{x \oplus b}$

Consequence: Assuming $V = L$, F_{ω_1} is not Borel graphable

2.2 F_{ω_1} is Borel graphable

Thm (Arant) Assume ~~$V=L$~~ all reals are in L. Then F_{ω_1} is not Borel graphable

Thm (Arant-Kechris-L.) Assume there is a real not in L. Then F_{ω_1} is Borel graphable.

F_{ω_1} Borel graphable $\Leftrightarrow$ there is a non-constructible real

Cor Borel graphability is not Π^1_3
Not preserved downward

Proof idea: ① Reduce to the case where there is a real not in L and $\omega_1^L = \omega_1$
Using absoluteness (thanks Gabe Goldberg)

② Use Kumabe-Slaman forcing

Assume: There is $a \in 2^\omega$ s.t. $a \notin L$ and $\omega_1^L = \omega_1$

Goal: Show F_{ω_1} is Borel graphable

The graph: Set x and y adjacent if $x \oplus y \oplus a$ computes a witness that $\omega_1^x = \omega_1^y$

$\Delta^1_1(a)$

Enough to show: Given $x, y \in 2^\omega$ with $\omega_1^x = \omega_1^y$, we can find z s.t.

① $\omega_1^z = \omega_1^x$

② $a \oplus z$ computes enough information

computing $\mathcal{O}^x, \mathcal{O}^y, \mathcal{O}^z$ is enough

Perfect tool to build z: Kumabe-Slaman forcing

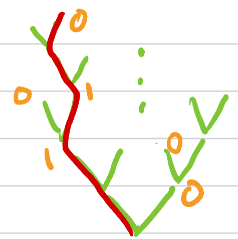
2.3 Kumabe-Slaman forcing

Kumabe-Slaman generic: partial labelling of $2^{<\omega}$

i.e. $f : 2^{<\omega} \to \{0, 1, \bot\}$ — $\bot$: "no label"

Computational interpretation: KS-generic g + $x \in 2^{\omega}$ $\rightsquigarrow$ $g(x) \in 2^{\leq \omega}$

read off the bits along x

$x = 001011\ldots$ $\qquad$ $g =$  $\rightsquigarrow$ $g(x) = 110\ldots$

Key property: Given ctbl model M and $a \notin M$, we can build g KS-generic over M s.t. $g(a)$ encodes essentially any information

even $\mathcal{O}^g$!!

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even $\mathcal{O}^g$!!

Thm (L.-Siskind) If g is KS-generic over M then for every $x \in M$, $\omega_1^{g \oplus x} = \omega_1^x$

KS-forcing preserves admissibility

Recall: Fix $a \notin L$. Assume $\omega_1^L = \omega_1$

Given x, y s.t. $\omega_1^x = \omega_1^y$, want z s.t.

① $\omega_1^z = \omega_1^x$

② $z \oplus a$ computes $\mathcal{O}^x$, $\mathcal{O}^y$, $\mathcal{O}^z$

pf sketch Take $z_0 \in L$ s.t. $\omega_1^{z_0} = \omega_1^x$

α big enough that $z_0 \in L_\alpha$ & $L_\alpha \models ZFC$

Build g KS-generic over L_α s.t. $g(a)$ encodes $\mathcal{O}^x$, $\mathcal{O}^y$ and $\mathcal{O}^{z_0 \oplus g}$

Take $z = z_0 \oplus g$. Thm $\Rightarrow \omega_1^z = \omega_1^{z_0} = \omega_1^x$

③ Two questions

Typical pattern: Want to show E Borel graphable

$\Downarrow$

show arbitrary information can be encoded into every E equiv. class

Def An analytic equivalence relation E on X has the **Borel coding property** if there is a Borel function $f: X \to 2^\omega$ s.t. for all $x \in X$ and $a \in 2^\omega$ there is $y \mathrel{E} x$ s.t. $f(y) \geq_T a$

Each equiv. class has elts encoding arbitrarily much info.

Comment F_{ω_1} has the Borel coding property $\Leftrightarrow$ $\exists$ real not in L

Question Borel coding property $\Rightarrow$ Borel graphable?

Observation: Every Borel graphing so far has diameter 2

Is this necessary?

<u>Question</u> Suppose E is Borel graphable. Must E have a graphing of diameter 2?

④ Polish group actions

Many interesting analytic equivalence relations arise from Polish group actions

Def Suppose Γ is a Polish group acting continuously on a Polish space X. The orbit equivalence relation of $\Gamma \curvearrowright X$, denoted E_Γ^X, is defined by

$$x \, E_\Gamma^X \, y \iff \exists g \in \Gamma \quad g \cdot x = y$$

Def A Polish group Γ is graphic if for every Polish group action $\Gamma \curvearrowright X$, E_Γ^X is Borel graphable

Question Which Polish groups are graphic?
All of them?

4.1 Facts about graphic groups

Γ graphic $\iff$ E_Γ^X is always Borel graphable

The class of graphic groups is closed under:

1. Finite products
2. Semidirect products
3. Quotients
4. Countable-index closed subgrps
5. Countable unions
6. Countable products

Other facts:

1. S_∞ is graphic

   By a graph of diameter 4 (!)

2. For $\Gamma \curvearrowright X$ Polish grp action, E_Γ^X has Borel coding property $\Rightarrow$ E_Γ^X Borel graphable

3. To show every Polish grp graphic, enough to show it for all totally disconnected grps

   B/c there is a projectively universal grp which is totally disconnected

Recall: E has Borel coding prop. if $\exists f: X \to 2^\omega$ s.t. for all $x \in X$, $a \in 2^\omega$ $\exists\, y E x$ s.t. $f(y) \geq_T a$

4.2 Connected groups

Fact To show every Polish group is graphic, it is enough to show it for all totally disconnected groups

Thm (Arant-Kechris-L.) All connected Polish groups are graphic 🤔

Fact For $\Gamma \curvearrowright X$ a Polish group action, E_Γ^X has Borel coding property $\Rightarrow$ E_Γ^X Borel graphable

Proof idea: Fix Γ connected, $\Gamma \curvearrowright X$ Polish grp action

For Thm

Can assume all orbits have size ≥ 2

Enough to show E_Γ^X has Borel coding property

Key idea: Code information into distances

Γ Connected Polish group

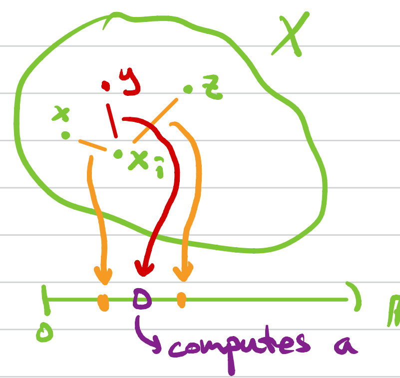
$\Gamma \curvearrowright X$ Polish group action w/ all orbits size ≥ 2

Goal: Show E_Γ^X has Borel coding property

<u>pf sketch</u> Fix $\{x_i\}_{i\in\mathbb{N}}$ ctbl dense set in X

$d(-,-)$ Polish metric on X

Define $f(x) = \bigoplus_{i\in\mathbb{N}} d(x, x_i)$



Given $x \in X$ and $a \in 2^\omega$

Pick $z \neq x$ in same orbit as x

and x_i s.t. $d(x, x_i) \neq d(z, x_i)$

connected

Consider $\Gamma \longrightarrow \mathbb{R}_{\geq 0}$

$g \longmapsto d(g\cdot x, x_i)$ continuous

$\Rightarrow$ Every pt in interval $[d(x,x_i), d(z,x_i)]$ is hit

Pick $e \in [d(x,x_i), d(z,x_i)]$ s.t. e computes a

$g \in \Gamma$ s.t. $d(g\cdot x, x_i) = e$. Set $y = g\cdot x$. $\Rightarrow f(y) \geq_T a$