① The Question

<u>Question</u> Which partial orders can be embedded in the Turing degrees in a Borel way?

More formally...

<u>Def</u> If A and B are written $A \leq_B B$ Borel binary relations on 2^ω, say A is Borel reducible to B if there is a Borel function $f: 2^\omega \to 2^\omega$ such that

$$\forall x,y \in 2^\omega \quad A(x,y) \Leftrightarrow B(f(x), f(y))$$

<u>Question, restated</u> Which Borel ~~partial orders~~ quasi-orders on 2^ω are Borel reducible to Turing reducibility?

Seems kind of unfair, since $\leq_T$ is not a partial order on 2^ω, but just a quasi-order

→ reflexive, transitive relation
= "partial order" where some elements are equivalent

quasi-orders

<u>Question, restated</u> Which Borel ~~partial orders~~ on 2^ω are Borel reducible to Turing reducibility?

In the Turing degrees, every element has at most countably many predecessors

The same must be true of any quasi-order which is reducible to $\leq_T$

Is that the only restriction?

<u>Def</u> A quasi-order $(Q, \leq)$ is called locally countable if for any $q \in Q$, the set $\{p \in Q \mid p \leq q\}$ is countable

<u>Question, final form</u> Is every locally countable Borel quasi-order on 2^ω Borel reducible to Turing reducibility?

② Why?

Why should we care about this question?

Two reasons

① Sacks's question: which partial orders embed into the Turing degrees?
This is the Borel version

② Kechris's conjecture: Every countable Borel equivalence relation is Borel reducible to Turing equivalence
This is a strengthening of Kechris's conjecture

contradicts Martin's conjecture (Part 1)

Sacks's question & Kechris's conjecture/Martin's conjecture are two of the most long-standing open questions about the global structure of the Turing degrees

2.1 Sacks's Question

Question: Does every locally countable, size-continuum partial order embed into the Turing degrees

→ no req't to be Borel

→ partial vs. quasi-order makes no difference here

What's Known

1. Every size ω_1 lcpo embeds
   - lcpo: locally countable partial order
   - So answer is "yes" assuming CH

2. Independent of ZF

3. Universal size continuum lcpo's exist
   - "Hereditarily a column of"

What's not known

Independent of ZFC?

2.2 Martin's conjecture

Part 1 only

Turing degrees

Basic idea: If $f: \mathcal{D}_T \to \mathcal{D}_T$ is a Borel function then either eventually f is constant or eventually $f(x) \geq_T x$

what does this mean?

Def A cone of Turing degrees is a set of the form $\{y \mid y \geq_T x\}$ for some fixed x

called the "base" of the cone

Def $f: 2^\omega \to 2^\omega$ is Turing invariant if
$$\forall x, y \in 2^\omega \quad x \equiv_T y \Rightarrow f(x) \equiv_T f(y)$$

Conjecture (Part 1 of Martin's conjecture) If $f: 2^\omega \to 2^\omega$ is a Turing invariant Borel function then either

① $\exists c$ s.t. $f(x) \equiv_T c$ for all x in some cone

or ② $f(x) \geq_T x$ for all x in some cone

super useful

uses Borel determinacy

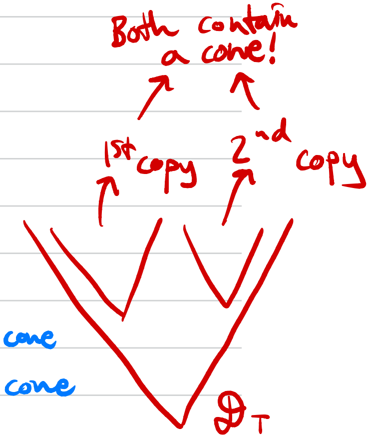
Thm (Martin) Every Borel set of Turing degrees either contains a cone or is disjoint from a cone

(2.3) Kechris's conjecture

Def An equivalence relation is countable if all its equivalence classes are countable

Conjecture (Kechris) Every countable Borel equivalence relation on 2^ω is Borel reducible to Turing equivalence

This contradicts part 1 of Martin's conjecture



pf Assume Kechris & Martin both true

Have Borel reduction f of $\mathcal{D}_T \times 2$ to $\mathcal{D}_T$

Get two functions $\mathcal{D}_T \to \mathcal{D}_T$

ctbl-to-one $\begin{cases} f_0 : \text{1st copy} \to \mathcal{D}_T \\ f_1 : \text{2nd copy} \to \mathcal{D}_T \end{cases}$

Martin: (conjecture) f_0 not constant on a cone $\Rightarrow f_0(x) \geq_T x$ on a cone

f_1 not constant on a cone $\Rightarrow f_1(x) \geq_T x$ on a cone

Martin: (thm) range(f_0) contains a cone

range(f_1) contains a cone

But any 2 cones have a cone in their intersection. Contradiction

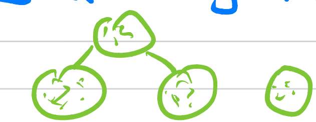
__Conjecture (Kechris)__ Every countable Borel equivalence relation on 2^ω is Borel reducible to Turing equivalence

__Original question__ Is every locally ctbl Borel quasi-order Borel reducible to Turing reducibility?

This is stronger than Kechris's conjecture

A countable equivalence relation __is__ a locally ctbl quasi-order

Typical ctbl equiv. rel.



Typical lcqo

A Borel reduction from $f: E \to \leq_T$ from a ctbl Borel equivalence relation (thought of as a Borel lcqo) to $\leq_T$ __is__ a Borel reduction of $E \to \equiv_T$

It sends different equivalence classes to __incomparable__ Turing degrees rather than just different Turing degrees

③ The Answer

<u>Question</u> Is every locally ctbl Borel quasi-order Borel reducible to Turing reducibility?

<u>Answer</u> No!

<u>Def</u> $f : 2^\omega \to 2^\omega$ is Turing-order-preserving if
$$\forall x, y \in 2^\omega \quad x \leq_T y \Rightarrow f(x) \leq_T f(y)$$

<u>Thm</u> (L.-Siskind) Part 1 of Martin's conjecture holds for all Borel Turing-order-preserving functions

Similar to Kechris $\Rightarrow$ ¬Martin, this shows 2 disjoint copies of $\leq_T$ cannot be Borel reduced to one copy

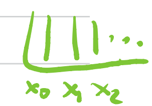
Natural follow-up question So which ones <u>are</u> Borel reducible to $\leq_T$?

(3.1) A Generalization

Question Which locally countable Borel quasi-orders are Borel reducible to Turing reducibility?

Example 1 $\leq_T \times 2$ is not reducible

Def A quasi-order $(Q, \leq)$ is countably directed if every countable subset of Q has an upper bound

Example Turing reducibility
$\{x_0, x_1, x_2, \ldots\}$ has upper bound $\bigoplus_{i \in \mathbb{N}} x_i =$ 

Example 2 $P_0 \sqcup P_1$ is not reducible
where P_0, P_1 are any uncountable, countably directed Borel lcqo's

Proof is very similar to thm on previous page

(3.2) 3 > 2

Question Which locally countable Borel quasi-orders are Borel reducible to Turing reducibility?

Example 3 (Higuchi - L.) Any locally countable Borel partial order of height 2 is reducible

Example 4 (Higuchi - L.) $\exists$ locally countable Borel partial order of height 3 not reducible

Def A quasi-order $(Q, \leq)$ has height n if the longest strict chain $x_1 \lneq x_2 \lneq \ldots$ has length n

= Q can be divided into n "layers"

Height 2

Warning Partial order of height 2 is much more restrictive than quasi-order of height 2

Proving example 3 for quasi-orders of height 1 would prove Kechris's conjecture

④ The Proof (sort of)

Thm (Higuchi-L.) Every locally countable Borel partial order of height 2 is Borel reducible to $\leq_T$

pf (sketch) $(Q, \leq)$ is a locally ctbl Borel partial order of height 2

$Q_1 = 1^{st}$ level $Q_2 = 2^{nd}$ level

By Lusin-Novikov, Q_1 & Q_2 are Borel

① Embed 1^{st} level

Classic thm: $\exists$ perfect set of mutually 1-generic reals

Map each $x \in Q_1$ to a diff. generic

② Embed 2^{nd} level

Map each $y \in Q_2$ to a generically chosen upper bound for images of its predecessors in Q_1

↳ can enumerate using Lusin-Novikov

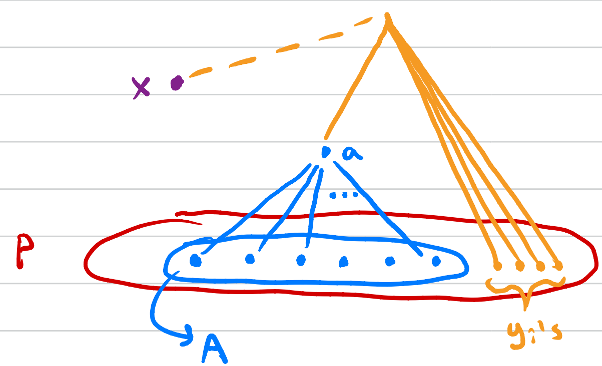
Why can't we just repeat step ② for a 3^{rd} level?

Common ingredient in all the non-reducibility proofs

A new "basis thm" for perfect sets

<u>Thm</u> (L.-Siskind) If $P \subseteq 2^\omega$ is a perfect set, $A \subseteq P$ is a countable dense subset of P and a is a real which computes every element of A then

$$\forall x\ \exists y_1, y_2, y_3, y_4 \in P\ (x \leq_T a \oplus y_1 \oplus y_2 \oplus y_3 \oplus y_4)$$



<u>Apology</u> I know this looks really technical ☹

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Used in combination with...

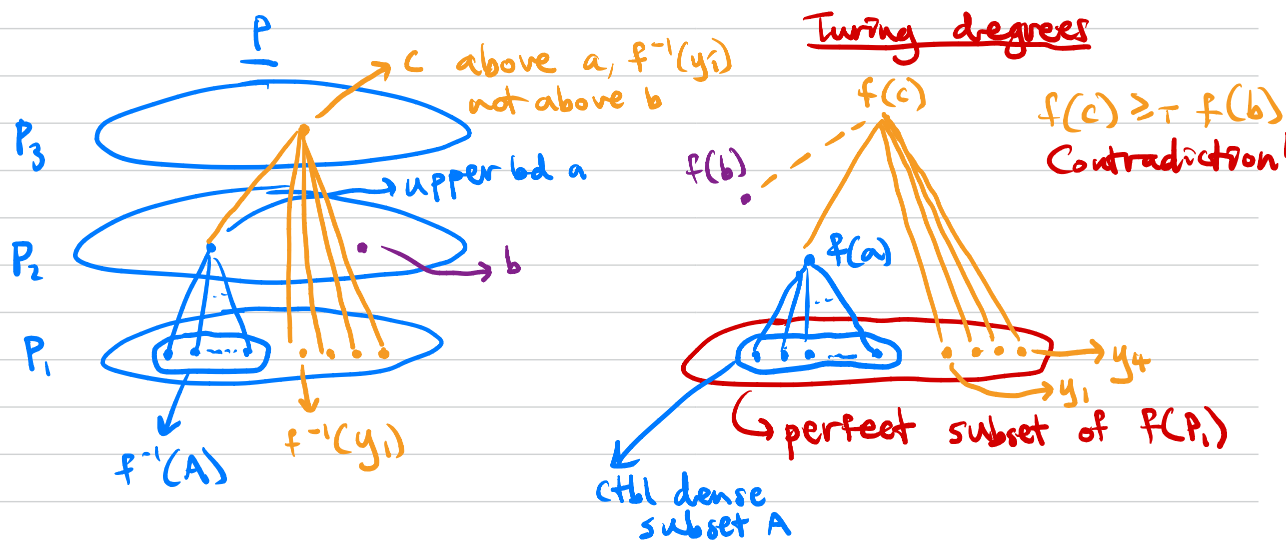
Thm (Suslin?) Every uncountable Σ^1_1 set contains a perfect set

$\Rightarrow$ if $f : 2^\omega \to 2^\omega$ Borel & ctbl-to-one & $A \subseteq 2^\omega$ Borel & uncountable then $f(A)$ contains a perfect set

Thm (Higuchi–L.) There is a locally ctbl Borel partial order of height 3 that is not Borel reducible to $\leq_T$

pf (sketch) levels of P: P_1 1st level
P_2 2nd level
P_3 3rd level

$f: P \to \leq_T$



Thm (Higuchi-L.) There is a locally ctbl Borel partial order of height 3 that is not Borel reducible to $\leq_T$

pf (sketch) Does a P with all necessary properties exist?

What did we use?

1. Borel, locally countable, height 3
2. Level 1 uncountable
3. Every ctbl subset of level 1 has an upper bound on level 2
4. For every finite subset of levels 1 & 2 & pt in level 2 not in that subset, there is an upper bd for the finite set in level 3 that is not above the pt

Can check that it is easy to construct P to satisfy these conditions

5 Lessons

One strategy to answer Sacks's question:

To embed $(P, \leq)$ in $\mathcal{D}_T$, use transfinite recursion on P to define embedding } "one-by-one" approach

This works if $|P| = \omega_1$

Does not work in general

<u>Thm</u> (Groszek–Slaman) It is consistent that there is a maximal independent set in $\mathcal{D}_T$ of size less than $|2^\omega|$

↳ If we choose badly, we could get stuck

Another approach

Embed 1st level of P as a perfect set of generics, 2nd level as generic upper bds, etc } "all at once" approach

Maybe this works for P well-founded? No!

As soon as you let a perfect set into the range of your embedding, you have lost

6 Questions

Two open questions

1. Is it provable in ZFC that every locally countable partial order on 2^ω of weight three embeds in $\mathcal{D}_T$?

   Probably!

   But if so, must use new techniques

2. Is every locally countable Borel quasi order of weight 1 Borel reducible to $\leq_T$?

   I think not

   Refuting this is easier than refuting Kechris's conjecture