MATH 54 SUMMER 2017, QUIZ 17

Find all eigenvalues and corresponding eigenspaces of the matrix A.

$$A = \begin{bmatrix} -6 & 3 & 0 \\ -6 & 3 & 0 \\ -3 & 3 & -3 \end{bmatrix}$$

First we will find the eigenvalues. To do this we need to find the characteristic polynomial:

$$\det \begin{bmatrix} -6-\lambda & 3 & 0 \\ -6 & 3-\lambda & 0 \\ -3 & 3 & -3-\lambda \end{bmatrix}.$$

Expanding along the last column gives us:

$$\begin{aligned} (-3-\lambda)\det\begin{bmatrix} -6-\lambda & 3 \\ -6 & 3-\lambda \end{bmatrix} &= (-3-\lambda)((-6-\lambda)(3-\lambda)-3(-6)) \\ &= (-3-\lambda)(-18+3\lambda+\lambda^2+18) \\ &= (-3-\lambda)(\lambda^2+3\lambda) \\ &= -\lambda(\lambda+3)^2. \end{aligned}$$

The roots of this polynomial are 0 (with multiplicity 1) and -3 (with multiplicity 2). So the eigenvalues are

$$\boxed{0 \text{ and } -3}$$

Now we will find the eigenspaces. First $E_0 = \mathrm{Null}(A - 0I) = \mathrm{Null}\, A$. Row reducing gives us

$$\begin{bmatrix} -6 & 3 & 0 \\ -6 & 3 & 0 \\ -3 & 3 & -3 \end{bmatrix} \xrightarrow{R2=R2+R1} \begin{bmatrix} -6 & 3 & 0 \\ 0 & 0 & 0 \\ -3 & 3 & -3 \end{bmatrix} \xrightarrow{R3=R3-\frac{1}{2}R1} \begin{bmatrix} -6 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 3/2 & -3 \end{bmatrix}$$

$$\xrightarrow{\text{Swap } R2 \text{ and } R3} \begin{bmatrix} -6 & 3 & 0 \\ 0 & 3/2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Solving the homogeneous equation for the above matrix gives us

$$\begin{aligned} &x_3 \text{ is free} \\ (3/2)x_2 - 3x_3 = 0 &\implies x_2 = 2x_3 \\ -6x_1 + 3x_2 = 0 &\implies x_1 = \frac{1}{2}x_2 = x_3 \end{aligned}$$

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Writing this in parametric form gives us

$$x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

and therefore

$$E_0 = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Now we find $E_{-3} = \text{Null}(A - (-3)I) = \text{Null}(A + 3I)$. Row reducing gives us

$$\begin{bmatrix} -3 & 3 & 0 \\ -6 & 6 & 0 \\ -3 & 3 & 0 \end{bmatrix} \xrightarrow{R2=R2-2R1} \begin{bmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ -3 & 3 & 0 \end{bmatrix} \xrightarrow{R3=R3-R1} \begin{bmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solving the homogeneous equation for the above matrix gives us

$$x_3 \text{ is free}$$
$$x_2 \text{ is free}$$
$$-3x_1 + 3x_2 = 0 \implies x_1 = x_2$$

Writing this in parametric form gives us

$$x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and therefore

$$E_{-3} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Let's check that the eigenvectors we found above really are eigenvectors of A with the claimed eigenvalues:

$$A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix}$$
$$A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$