# Math 54 Midterm, Summer 2017

Instructor: Patrick Lutz

July 17, 2017

Name: ______________________________

PLEDGE: I promise I will not cheat on this exam in any way.

Sign Here: ______________________________

INSTRUCTIONS: Answer each question in the space provided. If you run out of room, use the blank pages at the end. Good luck and, as it says on the cover of the *The Hitchhiker's Guide to the Galaxy*, Don't Panic.

| Question | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 4 | |
| 3 | 10 | |
| 4 | 4 | |
| 5 | 5 | |
| 6 | 6 | |
| 7 | 4 | |
| 8 | 4 | |
| 9 | 8 | |
| Total: | 50 | |

Don't turn over this page until you are told to do so.

1. (5 points) Are the following vectors linearly independent? If so, explain why. If not, find a nontrivial linear combination that is equal to zero.

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -3 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 3 \\ 2 \\ -1 \end{bmatrix}$$

2. (4 points) Find a basis for the eigenspace of the eigenvalue 3 of the matrix $A$ shown below.

$$A = \begin{bmatrix} 6 & -6 & 9 \\ 2 & -1 & 6 \\ -1 & 2 & 0 \end{bmatrix}$$

3. Let $\mathbb{P}_2 = \{a_2x^2 + a_1x + a_0 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ denote the vector space of all polynomials of degree at most two. Let $T\colon \mathbb{P}_2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}.$$

Let $\mathcal{B} = \{x-1, x^2-1, x^2+x\}$ and let $\mathcal{E}$ be the standard basis for $\mathbb{R}^3$.

(a) (4 points) Find the matrix of $T$ relative to the bases $\mathcal{B}$ and $\mathcal{E}$—i.e. find ${}_{\mathcal{E}}[T]_{\mathcal{B}}$.

(b) (4 points) Find the inverse of the matrix you found in part (a).

(c) (2 points) Find a degree two polynomial $p$ such that $p(0) = -2$, $p(1) = 2$, and $p(2) = 10$.

4. (4 points) Find a diagonal matrix similar to $A^{2017}$ where

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}.$$

5. (5 points) What is the determinant of the following matrix? Briefly justify your answer.

$$\begin{bmatrix} 1 & 7 & 8 & 1 & 2 & 3 \\ 2 & -9 & 81 & 2 & 7 & 0 \\ 3 & 4 & 7 & 3 & 7 & -1 \\ 4 & 1 & 1 & 4 & 1 & 1 \\ 5 & 7 & -3 & 5 & 13 & 788 \\ 6 & -1 & -2 & 6 & -4 & -5 \end{bmatrix}$$

6. (a) (2 points) If $A$ is a $3 \times 5$ matrix, what is the smallest possible value of $\dim(\operatorname{Null} A)$? You do not need to explain your reasoning.

   (b) (1 point) Give an example of a $3 \times 5$ matrix $A$ where $\dim(\operatorname{Null} A)$ is as small as possible.

   (c) (2 points) If $A$ is a $3 \times 5$ matrix, what is the largest possible value of $\dim(\operatorname{Null} A)$? You do not need to explain your reasoning.

   (d) (1 point) Give an example of a $3 \times 5$ matrix $A$ where $\dim(\operatorname{Null} A)$ is as large as possible.

7. (4 points) Suppose $T\colon \mathbb{P}_2 \to \mathbb{P}_1$ is a linear transformation such that $T(5x+2) = 3x$ and $\ker(T) = \operatorname{span}\{x^2+2x\}$. Find two distinct polynomials $p$ and $q$ in $\mathbb{P}_2$ such that $T(p) = T(q) = 3x$.

8. (4 points) Suppose $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{C}$ are bases for $\mathbb{R}^2$ and $\mathcal{D}$ is the basis for $\mathbb{R}^2$ shown below. Find $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \quad \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \quad \underset{\mathcal{D}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

9. Mark each of the following true or false. If true, briefly explain why. If false, give a counterexample.

   (a) (2 points) For any $n \times m$ matrix $A$ and any $m \times p$ matrix $B$, if $\operatorname{Col} AB = \mathbb{R}^n$ then $\operatorname{Col} A = \mathbb{R}^n$.

   (b) (2 points) For any linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and any $n \times n$ matrix $A$, $A\mathbf{u}$ and $A\mathbf{v}$ are linearly independent.

   (c) (2 points) For any basis $\{\mathbf{u}, \mathbf{v}\}$ for $\mathbb{R}^2$ and any $n \times 2$ matrices $A$ and $B$, if $A\mathbf{u} = B\mathbf{u}$ and $A\mathbf{v} = B\mathbf{v}$ then $A = B$.

   (d) (2 points) For any $2 \times 2$ matrices $A$ and $B$, if $A$ and $B$ are invertible then $AB = BA$.

Scratch paper