Math 54 Final, Summer 2017

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Name: ______________________________

PLEDGE: I promise I will not cheat on this exam in any way.

Sign Here: ______________________________

INSTRUCTIONS: Answer each question in the space provided. If you run out of room, use the blank pages at the end. Good luck!

Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine."
–Sir Michael Atiyah

Question	Points	Score
1	2	
2	4	
3	4	
4	5	
5	4	
6	9	
7	4	
8	2	
9	16	
Total:	50	

Don't turn over this page until you are told to do so.

1. Carefully complete each of the following definitions.

   (a) (1 point) The vector $\mathbf{u}$ is in the span of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ if and only if

   Solution: for some $c_1, c_2, c_3 \in \mathbb{R}$, $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{u}$. Equivalently, $\mathbf{u}$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

   Common Mistakes: Some people wrote that c_1, c_2, c_3 should not all be 0. That phrase appears in the definition of linear independence, but is not correct here since $\mathbf{0}$ *is* in the span of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ (cf. the common mistakes of question 9 part (a)). I did not take off points for this mistake.

   Another common mistake was to use a definition that only makes sense in $\mathbb{R}^n$. For instance, writing "the system $[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3]\mathbf{x} = \mathbf{u}$ is consistent."

   (b) (1 point) The vector $\mathbf{v}$ in the vector space V is an eigenvector of the linear transformation $T\colon V \to V$ with eigenvalue 5 if and only if

   Solution: $T(\mathbf{v}) = 5\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$.

   Common Mistakes: Many people forgot to say that $\mathbf{v}$ cannot be the zero vector. I did not take off points for this mistake.

   For this question also, some people gave a definition that only makes sense in $\mathbb{R}^n$, such as "$\mathbf{v}$ is in $\mathrm{Null}(A - 5I)$ where A is the standard matrix of T."

2. (4 points) Recall that the general solution to the heat equation with the usual boundary values is

$$\sum_{n=1}^{\infty} a_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

Suppose $\beta = 1$ and $L = \pi$. Find a solution $u(x,t)$ such that

$$u(x,0) = 31\sin(301x) - \sin(567x) + 12\sin(1000x).$$

Solution: Since $\beta = 1$ and $L = \pi$, the general solution is

$$\sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx).$$

To solve the initial value problem, we need to plug $t = 0$ into the general solution and solve for the coefficients. In other words, we need to find coefficients a_n such that

$$\sum_{n=1}^{\infty} a_n \sin(nx) = 31\sin(301x) - \sin(567x) + 12\sin(1000x).$$

Normally we would use Fourier series to find these coefficients, but here that's not necessary. It is clear that a_{301} should be 31, a_{567} should be -1, a_{1000} should be 12, and all other coefficients should be zero. Using these coefficients, the solution is

$$31e^{-31^2t}\sin(301x) - e^{-567^2t}\sin(567x) + 12e^{-1000^2t}\sin(1000x)$$

3. Let $U = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

(a) (1 point) Find a matrix A such that $\mathrm{Col}(A) = U$.

Solution:

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}$$

(b) (2 points) Find a basis for $\mathrm{Null}(A^T)$.

Solution:

$$A^T = \begin{bmatrix} 3 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

To find the null space, we need to row reduce.

$$\begin{bmatrix} 3 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix} \xrightarrow{\text{Swap } R1 \text{ and } R2} \begin{bmatrix} -1 & 0 & 2 \\ 3 & 2 & 3 \end{bmatrix} \xrightarrow{R2 = R2 + 3R1} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 9 \end{bmatrix}$$
$$\xrightarrow{R2 = \frac{1}{9}R2} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2/9 & 1 \end{bmatrix}$$

Solving the homogeneous equation of the above matrix gives us

$$\begin{aligned} x_3 &\text{ is free} \\ \frac{2}{9}x_2 + x_3 = 0 &\implies x_2 = -\frac{2}{9}x_3 \\ -x_1 + 2x_3 = 0 &\implies x_1 = 2x_3 \end{aligned}$$

Putting this in parametric form gives us

$$x_3 \begin{bmatrix} 2 \\ -\frac{2}{9} \\ 1 \end{bmatrix}$$

and therefore one basis for $\mathrm{Null}(A^T)$ is

$$\left\{ \begin{bmatrix} 2 \\ -\frac{2}{9} \\ 1 \end{bmatrix} \right\}$$

Common Mistakes: The most common mistake on this problem was making an arithmetic error during the row reduction. I did take off a point for this since it is easy to check that the result is in the null space of A^T.

(c) (1 point) Find a basis for $U^\perp$.

Solution: Since $\mathrm{Col}(A)^\perp = \mathrm{Null}(A^T)$ and since A was chosen so that $U = \mathrm{Col}(A)$, the answer to this part is the same as the answer to the previous part. So one basis for $U^\perp$ is

$$\left\{ \begin{bmatrix} 2 \\ -\frac{2}{9} \\ 1 \end{bmatrix} \right\}$$

One person also noticed that taking the cross product of the two given vectors will produce a third vector orthogonal to both. Since U has dimension 2, $U^\perp$ has dimension 1 and so this third vector must form a basis for $U^\perp$. However, this solution only works in $\mathbb{R}^3$, whereas the method used above works in any dimension.

Common Mistakes: Some people gave an answer to this question that included vectors in $\mathbb{R}^2$. But since U is a subspace of $\mathbb{R}^3$, so is $U^\perp$.

Also, some people gave answers that included the columns of A. But no nonzero vector can ever be in a subspace and its orthogonal complement.

4. (5 points) Find the orthogonal projection of $\mathbf{v}$ on W.

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\} \qquad \mathbf{v} = \begin{bmatrix} 8 \\ 0 \\ 0 \\ 9 \end{bmatrix}$$

Solution: There are two ways to solve this problem.

First method: One method is to find an orthogonal basis for W using the Gram-Schmidt algorithm and then use this basis to calculate the orthogonal projection of $\mathbf{v}$ on W.

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be the basis for W given in the problem statement. We will use Gram-Schmidt on these two vectors to produce an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for W.

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 \\
\mathbf{v}_2 &= \mathbf{u}_2 - \text{proj}_{\text{span}\{\mathbf{v}_1\}}(\mathbf{u}_2) \\
&= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \right) \\
&= \mathbf{u}_2 - \left(\frac{10}{10} \mathbf{v}_1 \right) \\
&= \begin{bmatrix} 4 \\ 1 \\ 2 \\ 2 \end{bmatrix}
\end{aligned}$$

We can now calculate the projection of $\mathbf{v}$ on W:

$$\begin{aligned}
\text{proj}_W(\mathbf{v}) &= \text{proj}_{\text{span}\{\mathbf{v}_1\}}(\mathbf{v}) + \text{proj}_{\text{span}\{\mathbf{v}_2\}}(\mathbf{v}) \\
&= \frac{\mathbf{v} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{\mathbf{v} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 \\
&= \frac{-10}{10} \mathbf{v}_1 + \frac{50}{25} \mathbf{v}_2 \\
&= \begin{bmatrix} 7 \\ 0 \\ 5 \\ 6 \end{bmatrix}
\end{aligned}$$

Second method: The second method is to use the algorithm for finding least squares solutions. More precisely, let A be a matrix whose column space is W and let $\widehat{\mathbf{x}}$ be a least squares solution to $A\mathbf{x} = \mathbf{v}$ (recall that we can find the least squares solution by solving $A^T A \widehat{\mathbf{x}} = A^T \mathbf{v}$). Since $A\widehat{\mathbf{x}}$ is the projection of $\mathbf{v}$ on the column space of A, $A\widehat{\mathbf{x}}$ is the answer to the question.

Common Mistakes: The most common mistake was to neglect to first find an orthogonal basis for W. Another common mistake was to use an incorrect formula when computing orthogonal projections.

5. (4 points) Write the following system of first order linear ODEs in normal form and then find the general solution.

$$\begin{aligned} y_1'(t) &= 2y_1(t) + 2y_2(t) \\ y_2'(t) &= 2y_1(t) - y_2(t) \end{aligned}$$

Solution: Normal form:

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

To find the solution we need to find a basis of eigenvectors of the matrix $\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$.

Characteristic polynomial:

$$\det \begin{bmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = (2-\lambda)(-1-\lambda) - 4 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2).$$

The roots of the characteristic polynomial are -2 and 3, both with multiplicity 1 (so we only need to find one eigenvector for each).

Eigenvector with eigenvalue -2:

$$\begin{bmatrix} 2-(-2) & 2 \\ 2 & -1-(-2) \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \xrightarrow{R1 = \frac{1}{2}R1} \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \xrightarrow{R2 = R2 - R1} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

Solving the homogeneous equation of the above matrix gives us

$$\begin{aligned} &x_2 \text{ is free} \\ x_1 &= -\frac{1}{2}x_2. \end{aligned}$$

Arbitrarily choosing $x_2 = 2$ gives us the eigenvector

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Eigenvector with eigenvalue 3:

$$\begin{bmatrix} 2-(3) & 2 \\ 2 & -1-(3) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \xrightarrow{R2 = R2 + 2R1} \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$$

Solving the homogeneous equation of the above matrix gives us

$$\begin{aligned} &x_2 \text{ is free} \\ x_1 &= 2x_2. \end{aligned}$$

Arbitrarily choosing $x_2 = 1$ gives us the eigenvector

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Therefore the general solution to the ODE is

$$c_1 e^{-2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

In other words, the general solution is

$$\begin{aligned} y_1(t) &= -c_1 e^{-2t} + 2c_2 e^{3t} \\ y_2(t) &= 2c_1 e^{-2t} + c_2 e^{3t} \end{aligned}$$

6. For this problem, you may assume without checking that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ shown below are linearly independent.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(a) (2 points) Suppose that $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation such that

$$\begin{aligned} T(\mathbf{v}_1) &= 2\mathbf{v}_1 \\ T(\mathbf{v}_2) &= \mathbf{v}_1 + 2\mathbf{v}_2 \\ T(\mathbf{v}_3) &= 3\mathbf{v}_3 \\ T(\mathbf{v}_4) &= \mathbf{v}_1 \end{aligned}$$

Find a basis $\mathcal{B}$ and a matrix A such that A is the matrix of T in the basis $\mathcal{B}$—i.e. such that ${}_{\mathcal{B}}[T]_{\mathcal{B}} = A$.

Solution: There are many possible answers to this question (infinitely many in fact, because $\mathbb{R}^4$ has infinitely many bases), but one of them is much easier to compute than the others. We already know what T does to the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$, so let's pick those vectors as our basis. In other words, let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Since any set of 4 linearly independent vectors in $\mathbb{R}^4$ must span $\mathbb{R}^4$, $\mathcal{B}$ is actually a basis. Now let's find the matrix of T in this basis.

$$\begin{aligned} A = {}_{\mathcal{B}}[T]_{\mathcal{B}} &= \big[[T(\mathbf{v}_1)]_{\mathcal{B}} \quad [T(\mathbf{v}_1)]_{\mathcal{B}} \quad [T(\mathbf{v}_1)]_{\mathcal{B}} \quad [T(\mathbf{v}_1)]_{\mathcal{B}}\big] \\ &= \big[[2\mathbf{v}_1]_{\mathcal{B}} \quad [\mathbf{v}_1 + 2\mathbf{v}_2]_{\mathcal{B}} \quad [3\mathbf{v}_3]_{\mathcal{B}} \quad [\mathbf{v}_1]_{\mathcal{B}}\big] \\ &= \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Common Mistakes: A lot of people picked the same basis as the solution above, but instead of writing ${}_{\mathcal{B}}[T]_{\mathcal{B}}$, instead wrote the matrix ${}_{\mathcal{E}}[T]_{\mathcal{B}}$. One of the important concepts in this problem is that a linear transformation is completely determined by what it does to a basis, and so when we want to analyze a linear transformation we can pick whatever basis is most convenient. In this problem, we already know exactly what T does to the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_4$, so the basis consisting of those vectors is most convenient. A number of people missed this aspect of the problem.

(b) (1 point) Find an invertible matrix P such that PAP^{-1} is the standard matrix of T. You do not need to show that the matrix you find is invertible and you do not need to find its inverse.

Solution: Since $A = {}_{\mathcal{B}}[T]_{\mathcal{B}}$, the matrix P should be the change of basis matrix from $\mathcal{B}$ to the standard basis, $\mathcal{E}$. In other words,

$$ {}_{\mathcal{E}}[T]_{\mathcal{E}} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} \, {}_{\mathcal{B}}[T]_{\mathcal{B}} \, \underset{\mathcal{B}\leftarrow\mathcal{E}}{P} $$

so P should be $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$. And the change of basis matrix $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ just consists of the vectors in the basis $\mathcal{B}$, written in terms of the standard basis vectors (i.e. written as coordinate vectors in the basis $\mathcal{E}$). In other words,

$$ P = \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 4 & -1 & 1 & 1 \\ 6 & 3 & 1 & 0 \\ 8 & 3 & 1 & 0 \end{bmatrix} $$

Common Mistakes: On this part of the problem, many people tried to find eigenvectors of the matrix A. This was incorrect for two reasons. First of all, this part of the problem does not really have anything to do with T or A, it is really just asking you to write the change of basis matrix from the basis you choose in part (a) to the standard basis. So unless you choose a basis of eigenvectors in part (a), eigenvectors are not relevant to this part of the problem. Second of all, the linear transformation T in this question happens to not be diagonalizable (which doesn't matter for any part of this problem).

(c) (2 points) What is the kernel of T?

Solution: First we will find a basis for the null space of the matrix A. This gives us a basis for the kernel of T, written as coordinate vectors in the basis $\mathcal{B}$. Since A is already in REF, solving the associated homogeneous equation is easy:

$$\begin{aligned} x_4 &\text{ is free} \\ 3x_3 = 0 &\implies x_3 = 0 \\ 2x_2 = 0 &\implies x_2 = 0 \\ 2x_1 + x_2 + x_4 = 0 &\implies x_1 = -\frac{1}{2}x_4 \end{aligned}$$

Writing this in parametric form gives us

$$x_4 \begin{bmatrix} -1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and therefore a basis for $\mathrm{Null}(A)$ is

$$\left\{ \begin{bmatrix} -1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

But remember that A is the matrix for T *in the basis* $\mathcal{B}$. So $\ker(T)$ is actually spanned by the vector whose coordinate vector in $\mathcal{B}$ is the vector above. In other words,

$$\ker(T) = \operatorname{span}\left\{ -\frac{1}{2}\mathbf{v}_1 + \mathbf{v}_4 \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ -3 \\ -4 \end{bmatrix} \right\}$$

Common Mistakes: Many people did not realize that the vectors in $\mathrm{Null}(A)$ are the *coordinate vectors in the basis* $\mathcal{B}$ of the vectors in $\ker(T)$.

(d) (2 points) What is the dimension of the range of T?

Solution: By the rank theorem, $\dim(\ker(T)) + \dim(\mathrm{range}(T)) = 4$. Since we found in the previous part that $\dim(\ker(T)) = 1$, this means that the dimension of the range of T is 3.

Common Mistakes: Some people answered this question without giving any explanation or showing any work.

(e) (2 points) What are the eigenvalues of T?

Solution: Since A is upper triangular, its eigenvalues are the entries on the diagonal. And since the eigenvalues of a linear transformation are independent of the choice of basis (i.e. similar matrices have the same eigenvalues), the eigenvalues of A are the same as the eigenvalues of T. So the eigenvalues of T are 2 (with multiplicity 2, though you weren't required to say this), 3, and 0. The question does not ask for eigenvectors for these eigenvalues, but they are not hard to find. In particular, $\mathbf{v}_1$ is an eigenvector with eigenvalue 2, $\mathbf{v}_3$ is an eigenvector with eigenvalue 3, and as we found in part (c), $-\frac{1}{2}\mathbf{v}_1 + \mathbf{v}_4$ is an eigenvector with eigenvalue 0.

Common Mistakes: A number of people did not realize that 0 was an eigenvalue of T, despite finding a nontrivial element of $\ker(T)$ in part (c).

7. (4 points) Suppose the Fourier sine series of $f(x)$ on the interval $[0, \pi]$ is

$$\sum_{n=1}^{\infty} \frac{1}{2n^2} \sin(nx).$$

Find the function $g(x)$ in $\operatorname{span}\{\sin(x), \sin(2x), \sin(3x)\}$ such that

$$\int_0^{\pi} (f(x) - g(x))^2 \, dx$$

is as small as possible.

Solution: Consider the vector space $C([0, \pi])$ of functions on the interval $[0, \pi]$. Define an inner product space on this vector space by $\langle f_1(x), f_2(x) \rangle = \int_0^{\pi} f_1(x) f_2(x) \, dx$ (i.e. the inner product on this vector space that we used repeatedly for the last week of the semester). With this inner product, the question is asking us to find $g(x) \in \operatorname{span}\{\sin(x), \sin(2x), \sin(3x)\}$ that minimizes

$$\langle f(x) - g(x), f(x) - g(x) \rangle = ||f(x) - g(x)||^2.$$

This is something we have learned how to do in any inner product space: $g(x)$ should just be the orthogonal projection of $f(x)$ on $\operatorname{span}\{\sin(x), \sin(2x), \sin(3x)\}$. Since we have already seen in class that $\sin(x)$, $\sin(2x)$, and $\sin(3x)$ are orthogonal, this can be calculated as follows:

$$\begin{aligned} g(x) &= \operatorname{proj}_{\operatorname{span}\{\sin(x), \sin(2x), \sin(3x)\}}(f(x)) \\ &= \frac{\langle f(x), \sin(x) \rangle}{\langle \sin(x), \sin(x) \rangle} \sin(x) + \frac{\langle f(x), \sin(2x) \rangle}{\langle \sin(2x), \sin(2x) \rangle} \sin(2x) + \frac{\langle f(x), \sin(3x) \rangle}{\langle \sin(3x), \sin(3x) \rangle} \sin(3x) \end{aligned}$$

Now observe that the coefficients in the linear combination above are exactly the first three coefficients in the Fourier sine series for $f(x)$. In other words, the solution is

$$\begin{aligned} g(x) &= \frac{1}{2(1)^2} \sin(x) + \frac{1}{2(2)^2} \sin(2x) + \frac{1}{2(3)^2} \sin(3x) \\ &= \frac{1}{2} \sin(x) + \frac{1}{8} \sin(2x) + \frac{1}{18} \sin(3x) \end{aligned}$$

Common Mistakes: Many people tried calculating the sine series of $g(x)$ and/or $f(x) - g(x)$, but got stuck because they didn't know what $g(x)$ was. It is actually not too hard to calculate the Fourier series of either of these once you have found $g(x)$, but it is probably not helpful for solving the problem.

Also, some people did not explain their answer or gave explanations that contained incorrect statements.

8. (2 points) Let $\mathcal{B}$ be the basis for $\mathbb{R}^3$ shown below. You may assume without checking that it is an orthonormal basis. Find $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P}$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{14} \\ 1/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}, \begin{bmatrix} -4/\sqrt{42} \\ 5/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix} \right\}$$

Solution: Since we are given the vectors in $\mathcal{B}$ written in terms of the standard basis, it is easy to find $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$. It is

$$\begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{14} & -4/\sqrt{42} \\ 1/\sqrt{3} & 1/\sqrt{14} & 5/\sqrt{42} \\ -1/\sqrt{3} & 3/\sqrt{14} & 1/\sqrt{42} \end{bmatrix}.$$

However, the question does not ask for $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$; it asks for $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P}$. Since $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}^{-1}$, we need to invert the matrix above. Normally, this would be pretty annoying, but the matrix above is orthogonal, so its inverse is just its transpose. In other words, the answer is

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 2/\sqrt{14} & 1/\sqrt{14} & 3/\sqrt{14} \\ -4/\sqrt{42} & 5/\sqrt{42} & 1/\sqrt{42} \end{bmatrix}$$

Common Mistakes: Some people got $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P}$ and $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ mixed up. Many people tried to invert $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ using row reduction and were not successful (a few people actually did this successfully and arrived at the correct answer).

9. Mark each of the following true or false. If true, briefly explain why. If false, give a counterexample. This question has eight parts on two pages.

(a) (2 points) If A and B are $n \times n$ matrices and $\mathbf{u}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are vectors such that $\mathbf{u} = A\mathbf{v}_1 + B\mathbf{v}_2$ then $\mathbf{u}$ is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Solution: False. Note that this would be true if A and B were scalars instead of matrices—the point of this problem is that multiplying vectors by matrices works very differently than multiplying vectors by scalars. There are many possible counterexamples, some more complicated than others. Here is one:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $\mathbf{u} = A\mathbf{v}_1 + B\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Common Mistakes: Some people correctly said that this was false, but then gave an example where $\mathbf{u}$ actually was in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. The most common pitfall here was to choose A and B so that $\mathbf{u}$ is the zero vector. This is bad because the zero vector is in the span of any other set of vectors!

(b) (2 points) If A is an $n \times n$ matrix such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution then for every $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ is consistent.

Solution: True. Since $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, when we put A in RREF there is a pivot in every column (i.e. there are no free variables). Since A is square, this means there must also be a pivot in every row and therefore $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^n$.

Common Mistakes: Some people correctly said that the statement was true, but gave explanations that made no use of the fact that A was square. Since this statement can be false when A is not square, these explanations were incomplete.

(c) (2 points) If $\{\mathbf{u}, \mathbf{v}\}$ is a basis for $\mathbb{R}^2$, $\mathbf{w}$ is a vector in $\mathbb{R}^2$, and a, b, c, and d are real numbers such that $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$ and $\mathbf{w} = c\mathbf{u} + d\mathbf{v}$ then $a = c$ and $b = d$.

Solution: True. I accepted a wide variety of explanations for this problem. Here is one explanation: if $a\mathbf{u} + b\mathbf{v} = c\mathbf{u} + d\mathbf{v}$ then $(a - c)\mathbf{u} + (b - d)\mathbf{v} = \mathbf{0}$ and since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, this implies that $a - c = b - d = 0$.

(d) (2 points) Suppose A and B are $n \times n$ matrices such that $AB = BA$. If $\mathbf{v}$ is an eigenvector of B and $A\mathbf{v}$ is nonzero then $A\mathbf{v}$ is an eigenvector of B.

Solution: True. Let λ be the eigenvalue of the eigenvector $\mathbf{v}$ of B. Then we have

$$\begin{aligned} B(A\mathbf{v}) &= (BA)\mathbf{v} \\ &= (AB)\mathbf{v} \\ &= A(B\mathbf{v}) \\ &= A(\lambda\mathbf{v}) \\ &= \lambda(A\mathbf{v}) \end{aligned}$$

Since $A\mathbf{v}$ is nonzero, this means $A\mathbf{v}$ is an eigenvector of B with eigenvalue λ.

Common Mistakes: Several people claimed that A and B are similar, which is not necessarily true: similar matrices don't necessarily commute, and commuting matrices are not necessariy similar.

Some people also seemed to think the question was asking if $\mathbf{v}$ was an eigenvector of A. That is in general false and also not what the question is asking.

Some people also wrote things like $B\mathbf{v} = \lambda\mathbf{v} \implies B\mathbf{v}A = \lambda\mathbf{v}A$. But in this class we never talked about multiplying a vector by a matrix on the right, so this doesn't make sense.

(e) (2 points) If A is an $n \times n$ matrix such that $A^4 = I_n$ then the only possible eigenvalues of A are 1 and -1.

Solution: False. The eigenvalues of A could also be i and $-i$. For instance if

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

then $A^4 = I_2$ but the eigenvalues of A are i and $-i$.

Notice that the above example is a matrix *with real entries*. Since the problem did not say A had to be real, I also accepted examples like

$$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}.$$

In fact, this was more or less the example given by everyone who got the problem correct—nobody gave an example of a real matrix with complex eigenvalues. ☹

Common Mistakes: Many people correctly observed that if λ is an eigenvalue of A then it must satisfy $\lambda^4 = 1$, but incorrectly concluded that this meant the only possibilities were 1 and -1.

Some people seemed to think the problem was saying that if $A^4 = I_n$ then A must have *both* 1 and -1 as eigenvaues. However, that is not what the statement above means.

(f) (2 points) If A is an $n \times n$ diagonalizable matrix whose only eigenvalues are 0 and 5 then $\mathrm{Col}(A)$ is equal to the eigenspace of the eigenvalue 5.

Solution: True. We need to show that everything in $\mathrm{Col}(A)$ is in the eigenspace of the eigenvalue 5 and also that everything in the eigenspace of eigenvalue 5 is in $\mathrm{Col}(A)$.

Since A is diagonalizable, there is some basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of eigenvectors of A. Let's suppose that the first k of them have eigenvalue 5 and the rest have eigenvalue 0. Let $\mathbf{x}$ be any vector in $\mathbb{R}^n$. Then for some $a_1, \ldots, a_n$, $\mathbf{x} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n$. Since the basis vectors are eigenvectors of A, we have

$$A\mathbf{x} = a_1 A\mathbf{v}_1 + \ldots a_n A\mathbf{v}_n = 5a_1\mathbf{v}_1 + \ldots + 5a_k\mathbf{v}_k.$$

This last vector is a linear combination of eigenvectors of A with eigenvalue 5 and hence it is in the eigenspace of the eigenvalue 5. Therefore any vector in $\mathrm{Col}(A)$ is also in the eigenspace of the eigenvalue 5.

On the other hand, if $\mathbf{x}$ is in the eigenspace of the eigenvalue 5 then by definition, $A\mathbf{x} = 5\mathbf{x}$. Therefore $A\left(\frac{1}{5}\mathbf{x}\right) = \mathbf{x}$ and so $\mathbf{x} \in \mathrm{Col}(A)$.

Several people wrote something along the following lines, which I also accepted: since A is diagonalizable, it is similar to a diagonal matrix whose diagonal contains only 0's and 5's. It is obvious that for such a matrix, its column space is equal to the eigenspace of eigenvalue 5. Since this fact does not depend on which basis we represent things in, this must also be true for the matrix A. By the way, the above argument is essentially a more precise version of this argument.

Common Mistakes: Many people gave incomplete explanations or proved it only for a specific 2×2 matrix.

(g) (2 points) If $f(x)$ is a solution to the ODE $y''(x) - 9y(x) = \sin^2(e^x)$ then so is $2e^{3x} + f(x)$.

Solution: True. There are several ways to see this. Some people simply plugged $2e^{3x} + f(x)$ into the differential equation and observed that it is satisfied. Another way to do this is as follows. Let T be the linear transformation $\frac{d^2}{dx^2} - 9I$. Then the problem is saying that $T(f(x)) = \sin^2(e^x)$. Observe that

$$T(2e^{2x}) = \frac{d^2}{dx^2}(2e^{3x}) - 9I(2e^{3x}) = 18e^{3x} - 18e^{3x} = 0$$

and therefore $2e^{3x} \in \ker(T)$. Since T is a linear transformation, this means $T(2e^{3x} + f(x)) = T(2e^{3x}) + T(f(x)) = 0 + \sin^2(e^x) = \sin^2(e^x)$.

Another way to say all of this is that a solution to a nonhomogeneous linear ODE plus a solution to the corresponding homogeneous ODE is also a solution to the original nonhomogeneous ODE.

(h) (2 points) There is no homogeneous, linear ODE for which $e^x \cos(x)$ and e^x are both solutions.

Solution: This is false. For instance, if the ODE is constant-coefficient then both functions will be solutions as long as the auxiliary equation has $1+i$, $1-i$ and 1 as roots. In other words, the auxiliary equation could be

$$(r - (1-i))(r - (1+i))(r - 1) = (r^2 - 2r + 2)(r - 1) = r^3 - 3r^2 + 4r - 2$$

and so one possible counterexample is

$$y''' - 3y'' + 4y' - 2y = 0.$$

Common Mistakes: Many people thought this problem was true because of a calculation involving the Wronskian of $e^x \cos(x)$ and e^x. Unfortunately this only tells us that there is no *second order* linear ODE with both functions as solutions. This is something that I completely failed to say clearly in class and that I probably further contributed to confusion about with a problem on the practice final. Therefore I have decided to count this exam out of 48 points. If you solved this problem correctly you still got points for it, but if not, it won't hurt your score on the final.

Also, some people who correctly said this statement was false did not realize that for $e^x \cos(x)$ to be a solution, both $1 + i$ *and* $1 - i$ must be roots of the auxiliary equation.