Math 54 Final, Summer 2017

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Name: ______________________________

Pledge: I promise I will not cheat on this exam in any way.

Sign Here: ______________________________

Instructions: Answer each question in the space provided. If you run out of room, use the blank pages at the end. Good luck!

> Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine."
> –Sir Michael Atiyah

Question	Points	Score
1	2	
2	4	
3	4	
4	5	
5	4	
6	9	
7	4	
8	2	
9	16	
Total:	50	

Don't turn over this page until you are told to do so.

1. Carefully complete each of the following definitions.

   (a) (1 point) The vector $\mathbf{u}$ is in the span of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ if and only if

   (b) (1 point) The vector $\mathbf{v}$ in the vector space V is an eigenvector of the linear transformation $T\colon V \to V$ with eigenvalue 5 if and only if

2. (4 points) Recall that the general solution to the heat equation with the usual boundary values is

$$\sum_{n=1}^{\infty} a_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

   Suppose $\beta = 1$ and $L = \pi$. Find a solution $u(x,t)$ such that

$$u(x,0) = 31\sin(301x) - \sin(567x) + 12\sin(1000x).$$

3. Let $U = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

(a) (1 point) Find a matrix A such that $\text{Col}(A) = U$.

(b) (2 points) Find a basis for $\text{Null}(A^T)$.

(c) (1 point) Find a basis for $U^\perp$.

4. (5 points) Find the orthogonal projection of $\mathbf{v}$ on W.

$$W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\} \qquad \mathbf{v} = \begin{bmatrix} 8 \\ 0 \\ 0 \\ 9 \end{bmatrix}$$

5. (4 points) Write the following system of first order linear ODEs in normal form and then find the general solution.

$$
\begin{aligned}
y_1'(t) &= 2y_1(t) + 2y_2(t) \\
y_2'(t) &= 2y_1(t) - y_2(t)
\end{aligned}
$$

6. For this problem, you may assume without checking that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ shown below are linearly independent.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(a) (2 points) Suppose that $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation such that

$$\begin{aligned} T(\mathbf{v}_1) &= 2\mathbf{v}_1 \\ T(\mathbf{v}_2) &= \mathbf{v}_1 + 2\mathbf{v}_2 \\ T(\mathbf{v}_3) &= 3\mathbf{v}_3 \\ T(\mathbf{v}_4) &= \mathbf{v}_1 \end{aligned}$$

Find a basis $\mathcal{B}$ and a matrix A such that A is the matrix of T in the basis $\mathcal{B}$—i.e. such that ${}_{\mathcal{B}}[T]_{\mathcal{B}} = A$.

(b) (1 point) Find an invertible matrix P such that PAP^{-1} is the standard matrix of T. You do not need to show that the matrix you find is invertible and you do not need to find its inverse.

(c) (2 points) What is the kernel of T?

(d) (2 points) What is the dimension of the range of T?

(e) (2 points) What are the eigenvalues of T?

7. (4 points) Suppose the Fourier sine series of $f(x)$ on the interval $[0, \pi]$ is

$$\sum_{n=1}^{\infty} \frac{1}{2n^2} \sin(nx).$$

Find the function $g(x)$ in $\operatorname{span}\{\sin(x), \sin(2x), \sin(3x)\}$ such that

$$\int_0^{\pi} (f(x) - g(x))^2 \, dx$$

is as small as possible.

8. (2 points) Let $\mathcal{B}$ be the basis for $\mathbb{R}^3$ shown below. You may assume without checking that it is an orthonormal basis. Find $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{14} \\ 1/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}, \begin{bmatrix} -4/\sqrt{42} \\ 5/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix} \right\}$$

9. Mark each of the following true or false. If true, briefly explain why. If false, give a counterexample. This question has eight parts on two pages.

   (a) (2 points) If A and B are $n \times n$ matrices and $\mathbf{u}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are vectors such that $\mathbf{u} = A\mathbf{v}_1 + B\mathbf{v}_2$ then $\mathbf{u}$ is in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

   (b) (2 points) If A is an $n \times n$ matrix such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution then for every $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ is consistent.

   (c) (2 points) If $\{\mathbf{u}, \mathbf{v}\}$ is a basis for $\mathbb{R}^2$, $\mathbf{w}$ is a vector in $\mathbb{R}^2$, and a, b, c, and d are real numbers such that $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$ and $\mathbf{w} = c\mathbf{u} + d\mathbf{v}$ then $a = c$ and $b = d$.

   (d) (2 points) Suppose A and B are $n \times n$ matrices such that $AB = BA$. If $\mathbf{v}$ is an eigenvector of B and $A\mathbf{v}$ is nonzero then $A\mathbf{v}$ is an eigenvector of B.

(e) (2 points) If A is an $n \times n$ matrix such that $A^4 = I_n$ then the only possible eigenvalues of A are 1 and -1.

(f) (2 points) If A is an $n \times n$ diagonalizable matrix whose only eigenvalues are 0 and 5 then $\mathrm{Col}(A)$ is equal to the eigenspace of the eigenvalue 5.

(g) (2 points) If $f(x)$ is a solution to the ODE $y''(x) - 9y(x) = \sin^2(e^x)$ then so is $2e^{3x} + f(x)$.

(h) (2 points) There is no homogeneous, linear ODE for which $e^x \cos(x)$ and e^x are both solutions.

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