Review

Suppose $S\colon \mathbb{R}^n \to \mathbb{R}^n$ and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations and let c be a real number. Prove that each of the following is a linear transformation.

1. The function $S + T\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $(S+T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$.
2. The function $cT\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $(cT)(\mathbf{v}) = cS(\mathbf{v})$.
3. $S \circ T$, the composition of S and T. (Recall that $S \circ T\colon \mathbb{R}^n \to \mathbb{R}^n$ is the function defined by $(S \circ T)(\mathbf{v}) = S(T(\mathbf{v}))$.)

Matrix Algebra

1. For each item below, either calculate the answer or explain why it is not defined.

$$A = \begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 6 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & -3 \\ 2 & 6 & 0 \\ 1 & 2 & 5 \end{bmatrix} \quad C \begin{bmatrix} -4 & 0 \\ 0 & 6 \\ 1 & 1 \end{bmatrix}$$

   (a) $A + B$
   (b) $B - 2I_3$
   (c) AB
   (d) A^3
   (e) B^2
   (f) BC
   (g) CA
   (h) CB

2. Suppose that A and B are 2×2 matrices such that $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ and $B\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$. Find a solution to $(A+B)\mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

3. Show that if $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation with standard matrix A and c is a real number then cA is the standard matrix of cT.

4. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation corresponding to rotation by $\pi/2$ radians counterclockwise and let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation corresponding to expansion by 3 in the x_2 direction (i.e. vertically).

   (a) Find the standard matrices of S and T. Let's call them A and B, respectively.
   (b) What is $T(S(\mathbf{e}_1))$? What about $T(S(\mathbf{e}_2))$?
   (c) What is the standard matrix of $T \circ S$? Try to describe what $T \circ S$ is doing geometrically.
   (d) What is the standard matrix of $S \circ T$? Try to describe what $S \circ T$ is doing geometrically.
   (e) What is AB? What is BA? How are they related to $T \circ S$ and $S \circ T$?

5. Find a 2×2 matrix A such that A is nonzero but $A^2 = 0$.

More Review

1. For what values of c is the linear transformation given below one-to-one? For what values of c is it onto?

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 5x_3 \\ 2x_1 + 4x_3 \\ 3x_1 + 6x_3 \\ x_1 + x_2 + cx_3 \end{bmatrix}$$