Review

1. Suppose that A is a 2×2 matrix with eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and corresponding eigenvalues 5 and -3 and that $\mathbf{u} = 2\mathbf{v}_1 - 8\mathbf{v}_2$. If $\mathbf{y}$ is a function from $\mathbb{R}$ to $\mathbb{R}^2$ that satisfies $\mathbf{y}'(t) = A\mathbf{y}(t)$ and $\mathbf{y}(0) = \mathbf{u}$, what is $\mathbf{y}(10)$? When t is very large, in approximately what direction is $\mathbf{y}(t)$ pointing?

Complex Eigenvalues and Nonhomogeneous Systems

1. Suppose $\mathbf{u}$ and $\mathbf{v}$ are vector-valued functions whose values are always real (not complex). Also, suppose $\mathbf{u}(t) + i\mathbf{v}(t)$ is a solution to $\mathbf{y}'(t) = A\mathbf{y}(t)$ where A is some matrix whose entries are all real numbers. Show that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are both solutions as well. (Hint: try writing down what it means for $\mathbf{u} + i\mathbf{v}$ to be a solution.)

2. Find the general solution of the following ODE.

$$\mathbf{y}'(t) = \begin{bmatrix} 5 & 1 \\ -8 & 1 \end{bmatrix} \mathbf{y}(t)$$

3. (If we have time) Find a solution to

$$\mathbf{y}'(t) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \mathbf{y}(t) + \begin{bmatrix} -4\cos(t) \\ -\sin(t) \end{bmatrix}$$

4. (a) Draw a picture of all the solutions to the following differential equation.

$$\mathbf{y}'(t) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{y}(t)$$

   (b) Draw a picture of all solutions to $\mathbf{y}'(t) = A\mathbf{y}(t)$ if A is $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ in the basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

5. (If we have time) Let A, B, and C, and D be the matrices shown below.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

   (a) Compute e^A
   (b) Compute e^{tA}
   (c) If $H = PAP^{-1}$, compute e^H and e^{tH} (hint: $tH = P(tA)P^{-1}$)
   (d) Compute e^{tC}
   (e) Using the fact that $e^{tD+tC} = e^{tD}e^{tC}$, compute e^{tB}

Definitions and Theorems

Definitions:

- (If we have time) Matrix exponential

Theorems:

- (If we have time) The fundamental matrix of $\mathbf{y}'(t) = A\mathbf{y}(t)$ is e^{tA} and the solution to the initial value problem $\mathbf{y}(0) = \mathbf{v}$ is $e^{tA}\mathbf{v}$.

Most important idea today: The most important thing today is to have an intuitive picture of what's going on when we solve a system of first order ODEs by finding eigenvectors of a matrix.
For instance, one intuitive idea is that solving the ODE $\mathbf{y}'(t) = A\mathbf{y}(t)$ by finding a basis of eigenvectors of A is equivalent to changing to a basis in which the ODE is "decoupled"—i.e. each differential equation in the system does not depend on the others.