Review

1. Suppose $X(t)$ is a fundamental matrix for the system $\mathbf{y}' = A\mathbf{y}$. Solve the initial value problem $\mathbf{y}' = A\mathbf{y}; \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$X(t) = \begin{bmatrix} e^{3t} & 2e^{7t} \\ 5e^{3t} & 7e^{7t} \end{bmatrix}$$

2. Write a system of first order linear ODEs that is equivalent to $\mathbf{y}'(t) = A\mathbf{y}(t)$ and then find all solutions to the system.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

The Hero Returns

1. Find the general solution to the following ODE. (Hint: the eigenvalues are -2 and 1).

$$\mathbf{y}'(t) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \mathbf{y}(t)$$

2. Solve the following initial value problem, where A is the matrix from the previous question.

$$\mathbf{y}'(t) = \mathbf{y}(t); \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

3. What is the long-term behavior of the solution you found in the previous question? As in, when t is very large, what is the approximate value of the solution?

4. Suppose that A is a 2×2 matrix with eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and corresponding eigenvalues 3 and 5 and that B is some invertible 2×2 matrix. Find the general solution of each of the following ODEs.

   (a) $\mathbf{y}' = A\mathbf{y}$
   (b) $\mathbf{y}' = A^2\mathbf{y}$
   (c) $\mathbf{y}' = A^{-1}\mathbf{y}$
   (d) $\mathbf{y}' = -3A\mathbf{y}$
   (e) $\mathbf{y}' = (-3A + 2I_2)\mathbf{y}$
   (f) $\mathbf{y}' = BAB^{-1}\mathbf{y}$

5. Find the general solution of the following ODE.

$$\mathbf{y}'(t) = \begin{bmatrix} 5 & 1 \\ -8 & 1 \end{bmatrix}$$

Definitions and Theorems

Definitions:

- Fundamental matrix
- Wronskian of vector-valued functions.

Theorems:

- If $\mathbf{u}$ is an eigenvector of A with eigenvalue r then $\mathbf{y}(t) = e^{rt}\mathbf{u}$ is a solution to $\mathbf{y}'(t) = A\mathbf{y}(t)$.

Most important idea today: To find solutions to $\mathbf{y}'(t) = A\mathbf{y}(t)$, find eigenvectors of A. Moral: any time you see a linear transformation, its eigenvectors are probably important!!