Review

1. True or false: if $\mathbf{u}$ and $\mathbf{v}$ are unit vectors in $\mathbb{R}^n$ such that $||\mathbf{u}-\mathbf{v}|| = \sqrt{2}$ then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
2. Suppose a least squares solution to $A\mathbf{x}=\mathbf{b}$ is $\mathbf{v}$. What is $\text{proj}_{\text{Col}\,A}(\mathbf{b})$? Is $\mathbf{b}$ in $\text{Col}(A)$?
$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
3. If A has orthonormal columns, what is the least squares solution to $A\mathbf{x}=\mathbf{b}$? (Hint: what is A^TA?)

Inner Product Spaces

1. Each of the following are not inner product spaces. For each one, state which of the properties of inner product spaces it does not satisfy and give an example to show it does not satisfy that property.
   (a) $\mathbb{P}_2$ with $\langle p, q\rangle = p(1)q(1) + p(2)q(2)$.
   (b) $\mathbb{R}^2$ with $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}\cdot(A\mathbf{y})$ where A is an invertible matrix. (Note: for some choices of matrix A, this actually is an inner product space. The matrices that make this an inner product space are called "positive definite.")
   (c) $C(\mathbb{R})$ with $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$.
   (d) $\mathbb{R}^2$ with $\langle \mathbf{x}, \mathbf{y}\rangle = (\mathbf{x}\cdot\mathbf{y})^2$.
2. Show that $\{1, \sin(x), \sin(2x)\}$ is an orthogonal set in the inner product space $C([-\pi,\pi])$ with inner product given by $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$. (Hint: $\sin(nx)\sin(mx) = \cos((n-m)x)/2 - \cos((n+m)x)/2$).
3. With the same inner product space as in the previous example, find the projection of the function $f(x) = x$ on the subspace $\text{span}\{1, \sin(x), \sin(2x)\}$. (Hint: you may need to use integration by parts.)
4. Find a polynomial p of degree at most one such that $(f(0)-p(0))^2 + (f(1)-p(1))^2 + (f(2)-p(2))^2$ is as small as possible, where $f(x) = x^3$.

Definitions and Theorems

Definitions and examples:

- Inner Product Space
- Important Examples of inner product spaces: $C([a,b])$ with the inner product $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$. $\mathbb{P}_n$ with the inner product $\langle p, q\rangle = p(a_0)q(a_0) + \ldots + p(a_n)q(a_n)$ where $a_0, \ldots, a_n$ are distinct real numbers. $\mathbb{R}^n$ with the inner product $\langle \mathbf{x}, \mathbf{y}\rangle = (A\mathbf{x})\cdot(A\mathbf{y})$ where A is an invertible $n \times n$ matrix.

- Length, unit vector, distance, orthogonal, orthogonal basis, orthogonal complement, orthonormal.
- Orthogonal projection.

Theorems:

- In a finite dimensional inner product space: for any subspace W and vector $\mathbf{v}$, there are unique vectors $\mathbf{w}$ in W and $\mathbf{z}$ in $W^{\perp}$ such that $\mathbf{v} = \mathbf{w} + \mathbf{z}$. The vector $\mathbf{w}$ is called the projection of $\mathbf{v}$ on W and it is the closest vector in W to $\mathbf{v}$ (using the definition of distance given by the inner product).
- If W is a subspace of a finite dimensional inner product space and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for W then for any $\mathbf{u}$ in W, $\mathbf{u} = \sum_{i=1}^{k} \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle} \mathbf{v}_i$. If $\mathbf{u}$ is not in W then this formula gives the projection of $\mathbf{u}$ on W.
- The Gram-Schmidt algorithm works in any finite dimensional inner product space.
- (Cauchy-Schwarz inequality) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space then $\langle \mathbf{v}, \mathbf{u} \rangle \leq ||\mathbf{v}|| \cdot ||\mathbf{u}||$.

Most important idea today: Everything we learned about dot products still works for any inner product space. (Caveat: actually, some modifications are needed for infinite dimensional inner product spaces, but everything still works in almost the same way.)