Review

1. Find the projection of $\mathbf{y}$ on $W = \text{span}\{\mathbf{u}, \mathbf{v}\}$.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

2. Suppose $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for a subspace W and that $\mathbf{u}$ is a vector in W such that

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 6 \\ 2 \end{bmatrix}$$

What is $\mathbf{v}_2 \cdot \mathbf{u}$?

3. **Challenge Problem:** Show that any rank r matrix can be written as a sum of r rank one matrices. Can such a matrix ever be written as a sum of less than r rank one matrices?

Projections and Orthogonal (Unitary) Matrices

1. Suppose U is an orthogonal (a.k.a. unitary) matrix. What is U^TU? What is UU^T?

2. Suppose the columns of U are orthonormal, but U is not square. What is U^TU? What about UU^T?

3. Suppose the columns of U are orthogonal. What can you say about U^TU?

4. Show that for any $n \times m$ matrix U with orthonormal columns and any vector $\mathbf{x} \in \mathbb{R}^n$, $||U\mathbf{x}|| = ||\mathbf{x}||$. Is this still true if the columns of U are just orthogonal?

Least Squares

1. Find all least squares solutions to both of the following systems.

$$\begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$$

2. Suppose a least squares solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{v}$. What is $\text{proj}_{\text{Col}\,A}(\mathbf{b})$?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3. If A has linearly independent columns, find the standard matrix for $\text{proj}_{\text{Col}\,A}$ in terms of A and A^T.

4. If A has orthonormal columns, what is the least squares solution to $A\mathbf{x} = \mathbf{b}$?

Definitions and Theorems

Definitions:

- Orthogonal matrix (Warning: Not what it sounds like). Also known as a unitary matrix (especially when complex numbers are involved).
- Least squares solution, least squares error, normal equations.

Theorems:

- If the columns of U are an orthonormal set then $U^T U = I$.
- If U is an orthogonal matrix then U is invertible and $U^{-1} = U^T$.
- If the columns of U are an orthonormal set (note that this is not as strong as saying that U is an orthogonal matrix since we are not requiring it to be square) then for any $\mathbf{x}$ and $\mathbf{y}$, $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$. In other words, these matrices preserve the lengths of vectors and the angles between them.
- If the columns of a matrix A are orthonormal then AA^T is the standard matrix of $\text{proj}_{\text{Col}\,A}$.
- The least squares solution of $A\mathbf{x} = \mathbf{b}$ can be found by solving $A^T A\widehat{\mathbf{x}} = A^T\mathbf{b}$.
- If the columns of A are linearly independent then $A^T A$ is invertible.

Most important idea today: Orthgoonal projections give us a way to find approximate solutions to inconsistent systems of linear equations.