Review

1. True or false: every set of linearly independent vectors is orthogonal.

2. Let $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Find the coordinate vector for $\mathbf{u}$ in the basis $\mathcal{B}$ of the subspace W.
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$$

3. (This one is kind of tricky.) Show that if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are any linearly independent vectors in $\mathbb{R}^n$ then there is some vector in $\mathbb{R}^n$ not orthogonal to $\mathbf{v}_1$ but orthogonal to all of $\mathbf{v}_2, \ldots, \mathbf{v}_k$. (Hint: If A is the matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_k$, can you rewrite this problem as a question about finding a solution to some equation involving A^T?)

4. **Challenge Problem:** If A is an $n \times m$ matrix of rank s and B is an $p \times r$ matrix of rank t, what is the dimension of the vector space $\{X \in M_{m\times p} \mid AXB = 0\}$?

Projections

1. If $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ then find a vector $\widehat{\mathbf{y}}$ in W and a vector $\mathbf{z}$ orthogonal to W such that $\mathbf{y} = \widehat{\mathbf{y}} + \mathbf{z}$. What is the distance from $\mathbf{y}$ to W?
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

2. Suppose W is a subspace of $\mathbb{R}^3$ and that $\mathbf{y}$ and $\text{proj}_W(\mathbf{y})$ are as shown below. What is $\text{proj}_{W^\perp}(\mathbf{y})$?
$$\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{proj}_W(\mathbf{y}) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

3. If W is a subspace of $\mathbb{R}^n$ and $\mathbf{y} \in W$, what is $\text{proj}_W(\mathbf{y})$? What about $\text{proj}_{W^\perp}(\mathbf{y})$?

4. If W is a subspace of $\mathbb{R}^n$, V is a subspace of W and $\mathbf{y} \in \mathbb{R}^n$, show that $\text{proj}_V(\text{proj}_W(\mathbf{y})) = \text{proj}_V(\mathbf{y})$.

5. If the columns of an $n \times m$ matrix U are orthogonal, what is U^TU?

6. Show that if the columns of an $n \times m$ matrix U are orthogonal then for any vector $\mathbf{x} \in \mathbb{R}^m$, $||U\mathbf{x}|| = ||\mathbf{x}||$.

Gram-Schmidt

1. Find an orthogonal basis for the column space of the following matrix.
$$\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$$

Definitions and Theorems

Definitions:

- Orthogonal Projection
- Orthogonal matrix (Warning: Not what it sounds like). Also known as a unitary matrix.

Theorems:

- For any subspace W of $\mathbb{R}^n$, every vector can be written in a unique way as a sum of a vector in W and a vector in $W^\perp$. The first vector in this sum is the closest vector in W to the original vector.
- If the columns of U are an orthonormal set then $U^TU = I$.
- If U is an orthogonal matrix then U is invertible and $U^{-1} = U^T$.
- If the columns of U are an orthonormal set (note that this is not quite the same as saying that U is an orthogonal matrix) then for any $\mathbf{x}$ and $\mathbf{y}$, $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.
- Given any basis for a subspace, there is an algorithm to find an orthogonal basis for the same subspace (the Gram-Schmidt algorithm).

Most important idea today: The idea of orthogonal projections and the way that they allow you to find orthogonal bases (via Gram-Schmidt).