Review

1. What is $\{\mathbf{0}\}^\perp$? (Assume $\mathbf{0}$ here is referring to the zero vector in $\mathbb{R}^n$.)
2. What is $(\mathbb{R}^n)^\perp$?
3. If W is a subspace of $\mathbb{R}^n$, what is $W \cap W^\perp$?
4. Show that for any vector $\mathbf{v}$ in $\mathbb{R}^n$, $\mathbf{v} \cdot \mathbf{v} \geq 0$. When is it exactly equal to zero?
5. **Super useful fact:** Show that if $\mathbf{u}$ is orthogonal to the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ then it is orthogonal to every vector in $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$.

Transpose and Orthogonal Complement

1. Show that for any $n \times m$ matrix A, $\text{Col}(A)^\perp = \text{Null}(A^T)$.
2. Show that for any $n \times m$ matrix A, $\dim(\text{Row}\, A) = \dim(\text{Col}\, A)$.
3. Show that if W is a subspace of $\mathbb{R}^n$, $\dim(W^\perp) = n - \dim(W)$. (Hint: think of W as the column space of some matrix.)
4. Show that $(W^\perp)^\perp = W$.

Orthogonal Basis

1. Suppose W is a subspace of $\mathbb{R}^n$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a basis for W. Also suppose $\mathbf{u}$ is in W and that $\mathbf{u} = c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m$. If $1 \leq i \leq m$, what is $\mathbf{u} \cdot \mathbf{v}_i$?
2. Solve the following system of linear equations without doing any row reduction. (Hint: the columns of the coefficient matrix are orthogonal to each other.)

$$\begin{array}{rcrcrcr} x_1 & + & 6x_2 & + & 2x_3 & = & 23 \\ 2x_1 & - & x_2 & + & x_3 & = & 1 \\ 3x_1 & & & - & 16x_3 & = & -29 \\ 4x_1 & - & x_2 & + & 11x_3 & = & 23 \end{array}$$

Definitions and Theorems

Definitions:

- Orthogonal
- Orthogonal complement
- Transpose
- Row Space
- Orthogonal Set, Orthogonal Basis
- Orthonormal Set, Orthonormal Basis
- Projection onto a subspace (i.e. $\text{proj}_W(\mathbf{u})$)

Theorems:

- If a vector is orthogonal to every vector in a list then it is also orthogonal to all vectors in the span of that list.
- $\text{Col}(A)^{\perp} = \text{Null}(A^T)$
- $\text{rank}(A) = \text{rank}(A^T)$
- An orthogonal set of *nonzero* vectors is linearly independent.
- For any subspace W of $\mathbb{R}^n$, every vector can be written in a unique way as a sum of a vector in W and a vector in $W^{\perp}$. The first vector in this sum is the closest vector in W to the original vector.

Most important idea today: Suppose you want to figure out how to write one vector as a linear combination of a list of vectors. When the vectors in the list are all orthogonal to each other, it is super easy to do.