Review

1. Suppose A is a 2×3 matrix such that

$$\text{Null } A = \text{span}\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}\right\} \quad \text{and} \quad A\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

How many solutions does $A\mathbf{x} = \begin{bmatrix}1\\0\end{bmatrix}$ have? If at most one, explain why. If more than one, find at least four solutions.

Enter the Hero

1. Which of the following are eigenvectors of $\begin{bmatrix}2 & 1\\-2 & 5\end{bmatrix}$? For each one that is an eigenvector, state the corresponding eigenvalue.

   (a) $\begin{bmatrix}1\\1\end{bmatrix}$ (b) $\begin{bmatrix}1\\2\end{bmatrix}$ (c) $\begin{bmatrix}0\\0\end{bmatrix}$ (d) $\begin{bmatrix}-3\\-6\end{bmatrix}$

2. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A, both with eigenvalue 5. True or false: $3\mathbf{v}_1 - \mathbf{v}_2$ an eigenvector of A. If true, find the corresponding eigenvalue. If false, give a counterexample.

3. (a) Why does the definition of eigenvector require that an eigenvector is nonzero (i.e. why would the definition be annoying or silly if the zero vector could count as an eigenvector)?

   (b) Why did the definition of eigenvector only talk about square matrices?

4. Suppose A, B, and P are $n \times n$ matrices, A and P are invertible, and that $\mathbf{v}$ is an eigenvector of A with eigenvalue λ_1 *and* an eigenvector of B with eigenvalue λ_2. Find an eigenvector and corresponding eigenvalue for each of the following matrices.

   (a) AB (b) A^{-1} (c) A^3 (d) A^{1000} (e) PAP^{-1}

5. For both matrices below, find the eigenvalues and corresponding eigenspaces.

$$\begin{bmatrix}1 & 2\\-1 & 4\end{bmatrix} \quad \begin{bmatrix}2 & 0 & 0 & 0\\0 & 2 & 0 & 0\\0 & 0 & 3 & 1\\0 & 0 & 0 & 4\end{bmatrix}$$

6. For a square matrix A, show that 0 is an eigenvalue of A if and only if A is not invertible.

7. What are the eigenvalues and corresponding eigenspaces of the zero matrix? What about the identity matrix?

8. Suppose $T\colon \mathbb{P}_2 \to \mathbb{P}_2$ is the linear transformation defined by $T(p) = x\frac{dp}{dx} + \frac{dp}{dx}$. Find the eigenvalues and eigenvectors of T.

9. **Challenge Problem:** True or false: for every degree n polynomial p with leading coefficient $(-1)^n$, there is some $n \times n$ matrix A so that p is the characteristic polynomial of A.