The Matrix of a Linear Transformation

1. For each of the following linear transformations, find a basis for the domain and codomain and write the matrix of the linear transformation relative to those bases.

   (a) $T\colon M_{2\times 2} \to M_{2\times 2}$ defined by $T(B) = AB$ where

   $$A = \begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix}$$

   (b) $T\colon \operatorname{span}\{\sin(x), \cos(x)\} \to \mathbb{R}^3$ defined by $T(f) = \begin{bmatrix} f(0) \\ f(\pi) \\ f(\pi/2) \end{bmatrix}$.

   (c) $T\colon \operatorname{span}\{\sin(x), \cos(x)\} \to \mathbb{R}$ defined by $T(f) = \int_0^\pi f(x)\,dx$.

2. Which of the linear transformations in the previous questions are one-to-one? Which ones are onto?

3. For each linear transformation from the previous question, find a basis for the kernel and for the range.

4. The following linear transformation is one-to-one and onto (you do not have to check this). Find its inverse.

   $$T\colon \mathbb{P}_1 \to \mathbb{R}^2$$
   $$T(p) = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix}$$

5. Explain why a degree one polynomial is uniquely determined by its values at 1 and 2. (Hint: look at the previous question.)

Change of Basis

For all the problems in this section, let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

1. (a) Is $\mathcal{B}$ a basis for $\mathbb{R}^2$? What about $\mathcal{C}$?

   (b) If $\mathbf{v}$ is a vector in $\mathbb{R}^2$ such that $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ then what is $\mathbf{v}$?

   (c) If $\mathbf{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ then what is $[\mathbf{u}]_{\mathcal{B}}$?

2. Try to find a matrix A such that for any $\mathbf{v} \in \mathbb{R}^2$, $A\mathbf{v} = [\mathbf{v}]_{\mathcal{B}}$.

3. If $\mathbf{v}$ is a vector in $\mathbb{R}^2$ such that $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ then what is $[\mathbf{v}]_{\mathcal{C}}$?

4. Try to find a matrix B such that for any $\mathbf{v} \in \mathbb{R}^2$, $B[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{C}}$.

5. Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix} \quad T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}.$$

   Find the matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

6. With T as in the previous question, find the standard matrix of T.