# Final Exam Review

**Disclaimer:** The following lists only mention material that we have covered since the second midterm and they are not guaranteed to be comprehensive or to accurately reflect what topics will be emphasized on the final exam.

**Computations that you should be able to do:**

- Find an orthogonal basis of eigenvectors for a real symmetric matrix.
- Find the SVD and reduced SVD of a matrix.
- Find the general solution of a constant-coefficient, linear, homogeneous higher order ordinary differential equation (including one whose characteristic polynomial has repeated roots or complex roots).
- Find a solution to an initial value problem for a constant-coefficient, linear, homogeneous higher order ordinary differential equation.
- Find the general solution to a system of first order linear ordinary differential equations (including if the matrix has complex eigenvalues but not including cases where the matrix is not diagonalizable).
- Find the solution to an initial value problem for a system of first order linear ordinary differential equations.
- Reduce a higher order linear ordinary differential equation to a system of first order linear ordinary differential equations.
- Do things like compute the norm of a vector or run the Gram-Schmidt process in an arbitrary inner product space.
- Find the Fourier series of a function.

**Fundamental definitions you should know by heart:**

- Symmetric matrix
- Singular values, right singular vectors, left singular vectors
- SVD and reduced SVD
- Characteristic polynomial of a higher order linear ODE
- Norm, orthogonality, etc in an arbitrary inner product space
- Fourier series of a function on $[-\pi, \pi]$

**Review Questions**

1. **Give an example or explain why no example exists:** Two $n \times m$ matrices $A$ and $B$ which have the same rank but such that there are no invertible matrices $C$ and $D$ such that $CAD = B$.
2. **Give an example or explain why no example exists:** A square matrix $A$ and a scalar $\lambda$ such that $\lambda^2$ is an eigenvalue of $A^2$ but $\lambda$ is not an eigenvalue of $A$.
3. **True or False:**
   (a) If $A$ and $B$ are $n \times n$ matrices and $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2$ are vectors in $\mathbb{R}^n$ such that $\mathbf{u} = A\mathbf{v}_1 + B\mathbf{v}_2$ then $\mathbf{u}$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Suppose $A$ and $B$ are $n \times n$ matrices such that $AB = BA$. If $\mathbf{v}$ is an eigenvector of $B$ and $A\mathbf{v}$ is nonzero then $A\mathbf{v}$ is an eigenvector of $B$.

(c) If $A$ is an $n \times n$ matrix such that $A^4 = I_n$ then the only possible eigenvalues of $A$ are 1 and $-1$.

(d) If $A$ is an $n \times n$ diagonalizable matrix whose only eigenvalues are 0 and 5 then $\mathrm{Col}(A)$ is equal to the eigenspace for $A$ corresponding to the eigenvalue 5.

(e) There is no homogeneous, linear ODE for which $e^t \cos(t)$ and $e^t$ are both solutions.

4. Let $\mathcal{B}$ be the basis for $\mathbb{R}^3$ shown below (which you may assume without checking is an orthonormal basis). Find $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P}$ where $\mathcal{E}$ denotes the standard basis for $\mathbb{R}^3$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -4/\sqrt{42} \\ 5/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{14} \\ 1/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \right\}$$

5. Let $V = \mathrm{span}\{\sin(x), \cos(x)\}$ and let $T \colon V \to V$ be the function defined by $T(f) = f'$ (i.e. $T(f)$ is the derivative of $f$). Find the eigenvalues of $T$.

6. Find three different $2 \times 5$ matrices whose nonzero singular values are 2 and 3. It's okay to express these matrices as products of other matrices.

7. Suppose $f \colon [-\pi, \pi] \to \mathbb{R}$ is a continuously differentiable function whose Fourier series is

$$3 + \sum_{n=1}^{\infty} \frac{1}{2n^2} \sin(nx).$$

(a) Find $\int_{-\pi}^{\pi} f(x)\, dx$.

(b) Find the Fourier series of $f(x) + \cos(x)$

(c) Find the Fourier series of $f(x)\cos(x)$

(d) Find the function $g(x)$ in $\mathrm{span}\{\sin(x), \sin(2x), \sin(3x)\}$ such that

$$\int_{-\pi}^{\pi} (f(x) - g(x))^2\, dx$$

is as small as possible.