How to tell if two matrices are similar?

$= \det(A - tI)$

① If $\chi_A(t) \neq \chi_B(t)$ then A & B not similar

but, can have $\chi_A(t) = \chi_B(t)$ even when A & B not similar (e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$)

② If A and B have the same eigenvalues but the eigenspaces have different dimensions then A and B are not similar

But even if they are all the same, it doesn't always mean A & B are similar

③ If A & B are both diagonalizable, they are similar if and only if they have the same eigenvals with same multiplicity

$A = PDP^{-1}$ $\qquad$ $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$\updownarrow$ same $\qquad$ eigenvals of A

$B = QDQ^{-1}$

$D = Q^{-1}BQ$

$A = PQ^{-1}BQP^{-1}$ $\Rightarrow$ A and B are similar

$\downarrow$

$(PQ^{-1})^{-1} = (Q^{-1})^{-1}P^{-1} = QP^{-1}$

Practice MT2 #3

4. $\mathbb{P}_2 = \{ a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}$

$T: \mathbb{P}_2 \to \mathbb{P}_2 \qquad T(p) = 2t \cdot \frac{d}{dt} p(t) + p(-1)$

Is there a basis B of $\mathbb{P}_2$ s.t. ${}_B[T]_B$ is diagonal?

$C = \{ t^2, t, 1 \}$

$${}_C[T]_C = \begin{bmatrix} | & | & | \\ [T(t^2)]_C & [T(t)]_C & [T(1)]_C \\ | & | & | \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$T(t^2) = 2t(2t) + (-1)^2 = 4t^2 + 1$

$T(t) = 2t \cdot 1 + (-1) = 2t - 1 = 0 \cdot t^2 + 2 \cdot t + (-1) \cdot 1$

$T(1) = 2t \cdot 0 + 1 = 1$

Diagonalize $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ & then translate the eigenvector basis back to $\mathbb{P}_2$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$3t^2 + 0t + 1$

$0 \cdot t^2 - t + 1$

$0 \cdot t^2 + 0 \cdot t + 1$

char. poly: $(4-\lambda)(2-\lambda)(1-\lambda)$

Eigenvalues: 4, 2, 1

$$\boxed{\{3t^2+1, -t+1, 1\}}$$

E_4: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & -1 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ $x_1 = 3x_3$, $x_2 = 0$, x_3 free $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

E_2: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$ $x_1 = 0$, $x_2 = -x_3$, x_3 free $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

E_1: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = 0$, $x_2 = 0$, x_3 free $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$_C[T]_C = A$

$_C[T]_C = \underset{C \leftarrow B}{P} \; \underline{_B[T]_B} \; \underset{C \leftarrow B}{P}^{-1}$

$A = P \underline{D} P^{-1}$

change of basis between two bases for $\mathbb{P}_2$

$\underset{C \leftarrow B}{P} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Goal: find this basis

$B = \{ p_1, p_2, p_3 \}$ s.t. $[p_1]_C = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow p_1 = 3t^2 + 1$

$\vdots$

$[p_3]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \longrightarrow \text{diagonalizable}$$

Eigenvector basis: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

A list of eigenvectors which do <u>not</u> form a basis for $\mathbb{R}^2$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \longrightarrow \text{diagonalizable}$$

Eigenvector basis: $\overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}^{\text{Eigenvecs. w/ eigenval 2}}, \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{\text{Eigenvec w/ eigenval 3}}$

Not a basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

lin. dep.

$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $P = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \underset{\text{std} \leftarrow B}{P}$

Check: $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

${}_B[T_A]_B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $B = \left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

Diagonalize A. Find P, and D diagonal, s.t. $A = PDP^{-1}$

① $\chi_A(t) = \det\begin{bmatrix} 1-t & -2 \\ 1 & 4-t \end{bmatrix} = (1-t)(4-t) - (-2)\cdot 1$

$= 4 - 5t + t^2 + 2$

$= t^2 - 5t + 6$

$= (t-2)(t-3)$

$A - 3I = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$x_1 = -x_2$, x_2 free $\Rightarrow x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigenvalues: 2, 3

② $E_2 = \text{Null}(A - 2I) = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

$\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_2 = R_2 + R_1} \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $x_1 = -2x_2$, x_2 free

$E_3 = \text{Null}(A - 3I) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

What is $A^{1000} = ?$

$$A^{1000} = PD\cancel{P^{-1}}\,\cancel{P}D\cancel{P^{-1}} \cdots\cdots \cancel{P}DP^{-1}$$

$$= P D^{1000} P^{-1}$$

easy to compute $\begin{bmatrix} 2^{1000} & 0 \\ 0 & 3^{1000} \end{bmatrix}$

If A (2×2) has complex eigenvals but real entries then

$$A = P\begin{bmatrix} a & -b \\ b & a \end{bmatrix}P^{-1} \text{ for some } a, b \in \mathbb{R}$$

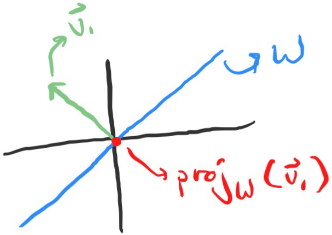
A^{1000} ?

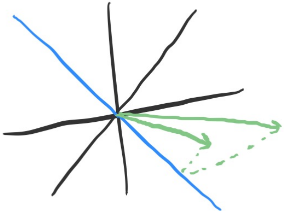
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \text{scaling} \cdot \text{rotation}$$

T/F If $\vec{v}_1, \dots, \vec{v}_k$ lin. ind. & W a subspace

then $\text{proj}_W(\vec{v}_1), \dots, \text{proj}_W(\vec{v}_k)$ lin. ind.

False





Change of Basis

Suppose $B = \{\vec{b}_1 \dots \vec{b}_n\}$ and $C = \{\vec{c}_1 \dots \vec{c}_n\}$ are bases for the same vector space, V.

If $\vec{v} \in V$ and you are given $[\vec{v}]_B$, what is $[\vec{v}]_C$?

$$[\vec{v}]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Answer: $[\vec{v}]_C = \underset{C \leftarrow B}{P} [\vec{v}]_B$

where $\underset{C \leftarrow B}{P} = \begin{bmatrix} [\vec{b}_1]_C & \dots & [b_n]_C \end{bmatrix}$

Suppose $\vec{b}_1 = x_{11} \vec{c}_1 + x_{12} \cdot \vec{c}_2 + \dots + x_{1n} \cdot \vec{c}_n$

$\vdots$

$\vec{b}_n = x_{n1} \cdot \vec{c}_1 + x_{n2} \cdot \vec{c}_2 + \dots + x_{nn} \cdot \vec{c}_n$

$$\begin{bmatrix} x_{11} & x_{21} & \dots & x_{n1} \\ \vdots & & \vdots & \\ x_{n1} & \dots & \dots & x_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$\vec{v} = a_1 \cdot \vec{b}_1 + a_2 \cdot \vec{b}_2 + \dots + a_n \cdot \vec{b}_n = a_1(x_{11} \cdot \vec{c}_1 + x_{12} \vec{c}_2 + \dots + x_{1n} \cdot \vec{c}_n)$

$+ \dots + a_n(x_{n1} \cdot \vec{c}_1 + \dots + x_{nn} \cdot \vec{c}_n)$

$= (a_1 \cdot x_{11} + a_2 \cdot x_{21} + \dots + a_n \cdot x_{n1}) \vec{c}_1 + \dots + (a_1 \cdot x_{1n} + \dots + a_n \cdot x_{nn}) \vec{c}_n$

① $\underset{D\leftarrow C}{P}\ \underset{C\leftarrow B}{P} = \underset{D\leftarrow B}{P}$

$\underset{D\leftarrow C}{P}\ \underset{C\leftarrow B}{P}\ [\vec{v}]_B = \underset{D\leftarrow C}{P}\ [\vec{v}]_C = [\vec{v}]_D$

② $\underset{C\leftarrow B}{P}^{-1} = \underset{B\leftarrow C}{P}$

③ If B is a basis for $\mathbb{R}^n$ $\quad B = \{\vec{b}_1 \dots \vec{b}_n\}$

then $\underset{std\leftarrow B}{P} = \begin{bmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{bmatrix}$

$[\vec{v}]_{std} = \vec{v}$

practice MT2 #2

2a $\mathbb{P}_2 = \{ a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}$

$T: \mathbb{P}_2 \to \mathbb{R}^2$ defined by $T(q) = \begin{bmatrix} q(2) \\ q(-3) \end{bmatrix}$

$B = \{ 1, t+1, t^2+t \}$ $\quad E = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \text{std}$

Find ${}_{\text{std}}[T]_B = \begin{bmatrix} [T(1)]_{\text{std}} & [T(t+1)]_{\text{std}} & [T(t^2+t)]_{\text{std}} \end{bmatrix}$

$= \begin{bmatrix} T(1) & T(t+1) & T(t^2+t) \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 & 6 \\ 1 & -2 & 6 \end{bmatrix}$

$T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$T(t+1) = \begin{bmatrix} 2+1 \\ -3+1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$T(t^2+t) = \begin{bmatrix} 2^2+2 \\ (-3)^2+(-3) \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$

Isomorphisms of vector spaces

If V and W are vector spaces, an isomorphism between V and W is a linear transformation $T: V \to W$ such that T is 1-to-1 and onto

"If V and W are isomorphic then they are the same in every way that matters"

Example: If $T: V \to W$ is an isomorphism then $\dim(V) = \dim(W)$

Suppose $\vec{v}_1, \ldots \vec{v}_n$ is a basis for V $\Rightarrow$ $\dim(V) = \dim(W) = n$

We can show $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ is a basis for W

Lin. ind.: If $a_1 T(\vec{v}_1) + \ldots + a_n T(\vec{v}_n) = \vec{0}$

$\Rightarrow T(a_1\vec{v}_1 + \ldots + a_n\vec{v}_n) = \vec{0}$ (linearity of T)

$\Rightarrow a_1\vec{v}_1 + \ldots + a_n\vec{v}_n = \vec{0}$ (b/c $T(\vec{0}) = \vec{0}$ & T is 1-to-1)

$\Rightarrow a_1 = a_2 = \ldots = a_n = 0$ (b/c $\vec{v}_1 \ldots \vec{v}_n$ lin. ind.)

An isomorphism is a linear transformation between two vector spaces

"Comparing vector spaces"

Similar matrices are two matrices A and B s.t. there is an invertible matrix P for which $A = PBP^{-1}$

"Comparing matrices/linear transformations"

Matrix w/ eigenvalue 0

A is $n \times n$

If A has 0 as an eigenvalue

$\iff$ there is a nonzero vector $\vec{v}$ s.t. $A\vec{v} = 0 \cdot \vec{v}$
$= \vec{0}$

$\iff$ there is some nonzero vector $\vec{v} \in Null(A)$

$\iff$ A is not invertible

$$E_0 = Null(A - 0 \cdot I) = Null(A)$$

Since A is square, A has free variables if and only if A has less than n pivots if and only if A has a row with no pivots

Example $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ free variable

all 0's

How to check if A is diagonalizable (A is $n \times n$)

1. Find the eigenvalues of A, $\lambda_1, \ldots, \lambda_k$
2. For each eigenvalue λ_i, find the dimension of $E_{\lambda_i} = \mathrm{Null}(A - \lambda_i I_n)$
3. If the dimensions sum to $n \Rightarrow$ diagonalizable
   otherwise $\Rightarrow$ not diagonalizable

"diagonalizable $\Longleftrightarrow$ there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A"

Assume A $n \times n$ diagonalizable

Show $(A^{-1})^2$ diagonalizable

$A = PDP^{-1}$ for P invertible & D diagonal

$$
\begin{aligned}
(A^{-1})^2 &= ((PDP^{-1})^{-1})^2 \\
&= (P^{-1^{-1}} D^{-1} P^{-1})^2 \\
&= (P D^{-1} P^{-1})^2 \\
&= P D^{-1} \overset{I}{\cancel{P^{-1} P}} D P^{-1} \\
&= P D^{-2} P^{-1}
\end{aligned}
$$

2 things that should be justified:

1. Why is D invertible?
2. Why is D^{-2} diagonal?