①

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} \cdots \vec{v}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \qquad T: \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad T(\vec{u}) = A\vec{u}$$

REF/pivots	systems of linear eq'n's	matrix eq'ns	span/lin. dep. of vectors	linear transformations
$[A\|b]$ has no pivot in last column when put in REF	$a_{11}\cdot x_1 + \cdots + a_{1n}\cdot x_n = b_1$ $\vdots$ $a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$ has a solution	$A\vec{x} = \vec{b}$ is consistent	$\vec{b} \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$	$\vec{b}$ is in the range of T
$[A\|b]$ has a pivot in every column except the last one in REF	$a_{11}x_1 + \cdots + a_{1n}\cdot x_n = b_1$ $\vdots$ $a_{m1}\cdot x_1 + \cdots + a_{mn}\cdot x_n = b_m$ has a unique sol'n	$A\vec{x} = \vec{b}$ has a unique solution	$\vec{b}$ can be written as a linear combination of $\vec{v}_1 \ldots \vec{v}_n$ in exactly 1 way	There is exactly 1 $\vec{x} \in \mathbb{R}^n$ s.t. $T(\vec{x}) = \vec{b}$

REF/pivots	systems of linear eq'ns	matrix eq'ns	vectors	linear transformations
A has a pivot in every col. in REF	$a_{11} \cdot x_1 + \dots + a_{1n} \cdot x_n = 0$ $\vdots$ $a_{m1} \cdot x_1 + \dots + a_{mn} \cdot x_n = 0$ has only the trivial sol'n	$A\vec{x} = \vec{0}$ has only the trivial sol'n	$\vec{v}_1, \dots, \vec{v}_n$ are linearly independent	T is 1-to-1
A has a pivot in every row in REF	$a_{11} \cdot x_1 + \dots + a_{1n} \cdot x_n = c_1$ $\vdots$ $a_{m1} \cdot x_1 + \dots + a_{mn} \cdot x_n = c_n$ has a solution for every $c_1, \dots, c_n \in \mathbb{R}$	$A\vec{x} = \vec{c}$ has a sol'n for every $\vec{c} \in \mathbb{R}^n$	$\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\} = \mathbb{R}^m$	T is onto

② a) For which values of c is T one-to-one? onto?

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+5z \\ 2x+4z \\ 3x+6z \\ x+y+cz \end{bmatrix}$$

standard matrix of T:

$$[T]_{std} = \left[T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right]$$

$$= \begin{bmatrix} 1 & 1 & 5 \\ 2 & 0 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & c \end{bmatrix} \xrightarrow[R_4 = R_4 - R_1]{\substack{R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1}} \begin{bmatrix} 1 & 1 & 5 \\ 0 & -2 & -6 \\ 0 & -3 & -9 \\ 0 & 0 & c-5 \end{bmatrix} \xrightarrow{R_2 = -\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & -3 & -9 \\ 0 & 0 & c-5 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 + 3R_2} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & c-5 \end{bmatrix}$$ REF

pivot if $c \neq 5$

one-to-one: if $c \neq 5$

(because then there is a pivot in every col. & if $c = 5$ then last col. is a free variable)

onto: never

(2) b) For which values of c are the following vectors linearly independent?

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 4 \\ 6 \\ c \end{bmatrix}$$

These are the columns of the standard matrix of the linear transformation T from part (a). So this is actually the same question as (a) and the answer is the same: they are linearly independent when $c \neq 5$.

c) For what values of c does $\begin{bmatrix} 1 & 1 & 5 \\ 2 & 0 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & c \end{bmatrix} \vec{x} = \vec{0}$ have a unique sol'n?

Same question again. As before: sol'n is unique for all values of c besides 5.

③ Check whether each function is a linear transformation. If so, find its standard matrix.

a) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$ 2 checks:

① $T\left(c \cdot \begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} cy \\ cx \end{bmatrix}$

$c \cdot T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = c \cdot \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} cy \\ cx \end{bmatrix}$ equal

② $T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 + y_2 \\ x_1 + x_2 \end{bmatrix}$

$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} y_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 + y_2 \\ x_1 + x_2 \end{bmatrix}$ equal

T is a linear transformation.

standard matrix: $[T]_{std} = \left[\, T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \;\; T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \,\right]$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b) $S: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+3 \\ y \end{bmatrix}$

2 checks:

① $S\left(c \cdot \begin{bmatrix} x \\ y \end{bmatrix}\right) = S\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} cx+3 \\ cy \end{bmatrix}$

$c \cdot S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = c \cdot \begin{bmatrix} x+3 \\ y \end{bmatrix} = \begin{bmatrix} cx+c \cdot 3 \\ cy \end{bmatrix}$

not equal if $c \neq 0$

S is not a linear transformation.

E.g. $4 \cdot S\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = 4 \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \end{bmatrix}$

$S\left(4 \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = S\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

not equal

④ a) Find all solutions

$$\begin{aligned} x_1 + 2x_2 + \quad\quad 4x_4 &= 0 \\ 2x_1 + 4x_2 + 5x_3 - 3x_4 &= 0 \\ 5x_1 + 10x_2 + \quad\quad 20x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 0 \\ 2 & 4 & 5 & -3 & 0 \\ 5 & 10 & 0 & 20 & 0 \end{array}\right] \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 5 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 = \frac{1}{5}R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -11/5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

free variables

$$\boxed{\begin{aligned} x_1 &= -2x_2 - 4x_4 \\ x_2 &\ \text{free} \\ x_3 &= 11/5\, x_4 \\ x_4 &\ \text{free} \end{aligned}}$$

Check:

$$(-2x_2 - 4x_4) + 2x_2 + 4x_4 = 0$$

$$2(-2x_2 - 4x_4) + 4x_2 + 5(11/5 \cdot x_4) - 3x_4 = -4x_2 - 8x_4 + 4x_2 + 11x_4 - 3x_4 = 0$$

$$5(-2x_2 - 4x_4) + 10x_2 + 20x_4 = -10x_2 - 20x_4 + 10x_2 + 20x_4 = 0$$

b) $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & 5 & -3 \\ 5 & 10 & 0 & 20 \end{bmatrix}$ write solution set of $A\vec{x} = \vec{0}$ in parametric form

Same matrix as part (a)

$$x_1 = -2\cancel{x_2}^{\,t} - 4\cancel{x_4}^{\,s}$$
$$x_2 \quad \cancel{\text{free}} = t$$
$$x_3 = (11/5)\cancel{x_4}^{\,s}$$
$$x_4 \quad \cancel{\text{free}} = s$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t-4s \\ t \\ (11/5)s \\ s \end{bmatrix} = t \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -4 \\ 0 \\ 11/5 \\ 1 \end{bmatrix}$$

for all $t, s \in \mathbb{R}$

c) Basis for Null(A). $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 11/5 \\ 1 \end{bmatrix} \right\}$

d) Basis for Col(A)

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & 5 & -3 \\ 5 & 10 & 0 & 20 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -11/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

become pivots in REF

pivot columns

One basis is:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \right\}$$

(5)

$$\det\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 0 & -1 & -6 \end{bmatrix}$$

Method 1:

$$\det \begin{array}{c} + \\ - \\ + \end{array}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 0 & -1 & -6 \end{bmatrix} = 1\cdot\det\begin{bmatrix} 2 & 5 \\ -1 & -6 \end{bmatrix} - 1\cdot\det\begin{bmatrix} 2 & 3 \\ -1 & -6 \end{bmatrix} + 0\cdot\det\begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= 1\cdot(2\cdot(-6) - 5\cdot(-1)) - 1\cdot(2\cdot(-6) - 3\cdot(-1))$$

$$= -12 + 5 - (-12 + 3)$$

$$= -12 + 5 + 12 - 3$$

$$= \boxed{2}$$

Method 2:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 0 & -1 & -6 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & -1 & -6 \end{bmatrix} \xrightarrow{\text{swap } R_1 \& R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -6 \\ 0 & 0 & 2 \end{bmatrix}$$

Multiplies det. by -1

upper triangular, so det. is product of diagonal

det. is $-1\cdot(1\cdot(-1)\cdot 2) = \boxed{2}$

⑥ True/False: If A is an $n \times m$ matrix then the set of sol'ns to $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^m$.

True (also called $Null(A)$)

3 checks:

① $\vec{0}$ is in the set?

Yes, $A \cdot \vec{0} = \vec{0}$

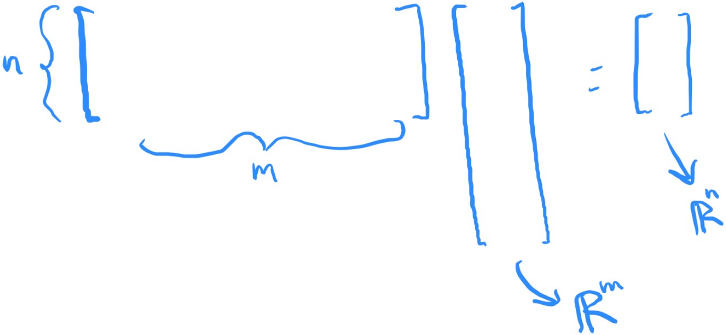
② if $\vec{x}$ and $\vec{y}$ are in the set, so is $\vec{x} + \vec{y}$?

Yes, $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$

③ if $\vec{x}$ is in the set and $c \in \mathbb{R}$ then $c \cdot \vec{x}$ is in the set?

Yes, $A(c \cdot \vec{x}) = c \cdot (A\vec{x}) = c \cdot \vec{0} = \vec{0}$

7. If A is an $n \times m$ matrix then $\mathrm{Col}(A)$ is a subspace of $\underline{\mathbb{R}^n}$ and $\mathrm{Null}(A)$ is a subspace of $\underline{\mathbb{R}^m}$.



⑧ Is it possible that $Col(A) = Null(A)$?

Yes. For example $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \xrightarrow{\text{swap } R_1 \& R_2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ RREF}$$

free variable

$Null(A)$: $x_1 = 0$, x_2 free $\rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

for any $t \in \mathbb{R}$.

So $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $Null(A)$

$Col(A)$: Take pivot columns of A as basis

So $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $Col(A)$

(9) Suppose $\text{Null}(A) = \text{Col}(B)$. What can you say about AB?

$AB = 0$

Reason: For any vector $\vec{v}$, $B\vec{v} \in \text{Col}(B)$

so $B\vec{v} \in \text{Null}(A)$. Thus $A(B\vec{v}) = \vec{0}$.

But $A(B\vec{v}) = (AB)\vec{v}$. So when you multiply any vector by AB you get $\vec{0}$ and thus AB must be the 0 matrix.

(10) Give an example of A, B not invertible such that AB invertible.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A not invertible because last column has no pivot.

B not invertible because last row has no pivot.

Can A & B be square?

No. $(AB)\vec{x} = \vec{0}$ has unique sol'n $\Rightarrow$ $B\vec{x} = \vec{0}$ has unique sol'n

So if AB invertible & B square, B must be invertible

(similar for A but use $Col(AB) = \mathbb{R}^n \Rightarrow Col(A) = \mathbb{R}^n$)

(11) A $n \times m$ matrix. How many sol'ns does $A\vec{x} = \vec{b}$ have if:

a) $\text{Null}(A) = \{\vec{0}\}$ $\quad \vec{b} \in \text{Col}(A)$ $\quad$ 1 solution.

$\Downarrow$ $A\vec{x} = \vec{0}$ has unique sol'n

$\Downarrow$ $A\vec{x} = \vec{b}$ consistent

b) $\text{Null}(A) \neq \{\vec{0}\}$ $\quad \vec{b} \in \text{Col}(A)$ $\quad$ infinitely many solutions.

$\Downarrow$ $A\vec{x} = \vec{0}$ has ∞ many sol'ns

$\Downarrow$ $A\vec{x} = \vec{b}$ consistent

c) $\vec{b} \notin \text{Col}(A)$ $\quad$ 0 solutions.

(12) If A is an $n \times m$ invertible matrix, what are $\mathrm{Null}(A)$ and $\mathrm{Col}(A)$?

$\mathrm{Null}(A) = \{\vec{0}\}$ $\quad$ $A\vec{x} = \vec{0}$ has only the trivial sol'n

$\mathrm{Col}(A) = \mathbb{R}^n$ $\quad$ Fo every $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a sol'n and so $\vec{b} \in \mathrm{Col}(A)$