PM #2 ① i) T/F: Every invertible matrix can be written as a product of elementary matrices.

True.

Matrix s.t. multiplying by it performs some elementary row operation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_2 = R_2 + 5R_1} \begin{bmatrix} a & b \\ c+5a & d+5b \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 5a+c & 5b+d \end{bmatrix}$$

elementary matrix

$$(A^{-1})^{-1} = A$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

$$[A \mid I_n] \xrightarrow{\text{row reduce}} [I_n \mid A^{-1}]$$

$$A^{-1} = B_k \cdots B_2 B_1 I_n = B_k \cdots B_2 \cdot B_1$$

① j) T/F

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; x_1 + x_2 + x_3 + x_4 \geq 0 \right\}$$

is a subspace of $\mathbb{R}^4$ False

3 checks

① $\vec{0} \in V$? Yes $0+0+0+0 \geq 0$

② Suppose $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in V$ $\begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_4 + y_4 \end{bmatrix} \in V$?

Yes $(x_1 + y_1) + \ldots + (x_4 + y_4)$

$= (x_1 + \ldots + x_4) + (y_1 + \ldots + y_4) \geq 0$

③ Suppose $\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} \in V$ $c \in \mathbb{R}$ $\begin{bmatrix} cx_1 \\ \vdots \\ cx_4 \end{bmatrix} \in V$?

No. Counterexample $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in V$ but $\begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \notin V$.

PM #2 2b Find A, B 2×2 s.t. AB invertible but BA is not invertible, or explain why no such matrices exist.

No such matrices exist.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Suppose A, B 2×2 s.t. AB invertible

We will show BA also invertible.

$(AB)\vec{x} = \vec{0}$ has only the trivial sol'n

$\Rightarrow$ $B\vec{x} = \vec{0}$ has only the trivial sol'n $\Rightarrow$ B invertible (b/c B square)

still true even if A, B not square

(if not, let $\vec{v} \neq \vec{0}$ s.t. $B\vec{v} = \vec{0}$.

Then $(AB)\vec{v} = A(B\vec{v}) = A\vec{0} = \vec{0}$

So $\vec{v}$ is a nontrivial sol'n to $(AB)\vec{x} = \vec{0}$)

$(AB)\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^2$

$\Rightarrow$ $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^2$ $\Rightarrow$ A is invertible

(For $\vec{b} \in \mathbb{R}^2$, let $\vec{v}$ be a sol'n to $(AB)\vec{x} = \vec{b}$

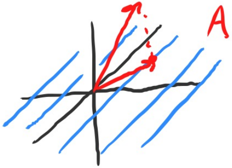
so $B\vec{v}$ is a solution to $A\vec{x} = \vec{b}$ b/c $A(B\vec{v}) = (AB)\vec{v} = \vec{b}$)

$(AB)\vec{x} = \vec{0}$ has only the trivial sol'n

$\Rightarrow$ $A\vec{x} = \vec{0}$ has only the trivial sol'n

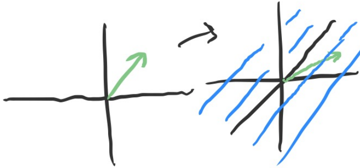
This is not true if A, B are not square

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

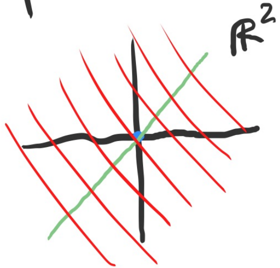


$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AB\vec{x}$

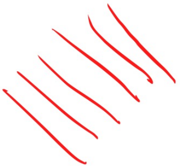


Subspaces

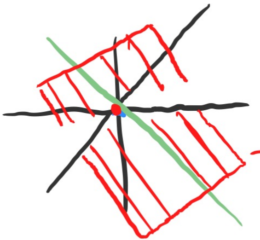


• = 0-dim subspace of $\mathbb{R}^2$

= 1 dim. subspace of $\mathbb{R}^2$



= 2 dim subspace of $\mathbb{R}^2$

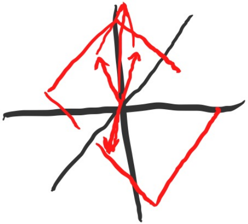


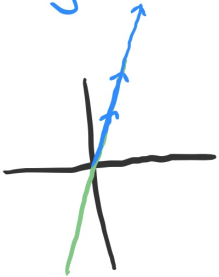
→ 2 dim. subspace of $\mathbb{R}^3$
(behaves "just like" $\mathbb{R}^2$)

Definition of a subspace of $\mathbb{R}^n$:

a set $V \subseteq \mathbb{R}^n$ is a subspace if all of the following hold:

1. $\vec{0} \in V$
2. For any $\vec{x}, \vec{y} \in V$, $\vec{x}+\vec{y} \in V$
3. For any $\vec{x} \in V$ and $c \in \mathbb{R}$, $c \cdot \vec{x} \in V$





Rank of A:

dimension of Col(A) = number of lin. ind. vectors that span Col(A)

→ size of a basis

$$A = \begin{bmatrix} 1 & 0 & \cancel{1} \\ 2 & 1 & \cancel{3} \\ 3 & 2 & \cancel{5} \end{bmatrix}$$

↓ redundant

Col(A) is spanned by

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$\text{rank}(A) = 2$

Algorithm to find rank(A): Put A into REF

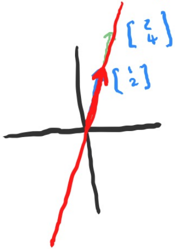
& rank = # of pivots

Suppose $V = \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ $\qquad \vec{v}_1 \ldots \vec{v}_n \in \mathbb{R}^m$

How to find a basis for V?

V is a subspace of $\mathbb{R}^m$

A basis for V is a set of vectors that span V and are <u>linearly independent</u>.



$$V = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$$

basis for V: $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

$\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$

$\left\{\begin{bmatrix} 3 \\ 6 \end{bmatrix}\right\}$

How to find a basis for $\text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$

$$= \text{Col}\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = A$$

algorithm: row reduce A to put in in REF & take columns of A corresponding to pivot columns in the REF matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a basis

A, B matrices such that AB invertible.

$(AB)\vec{x} = \vec{0}$ has a unique sol'n

$\Rightarrow$ $B\vec{x} = \vec{0}$ has a unique sol'n

Suppose $B\vec{x} = \vec{0}$ has a nontrivial solution.

This means there is $\vec{v} \neq \vec{0}$ s.t. $B\vec{v} = \vec{0}$.

Therefore
$$
\begin{aligned}
(AB)\vec{v} &= A(B\vec{v}) \\
&= A \cdot \vec{0} \\
&= \vec{0}
\end{aligned}
$$

$\vec{v}_1 \ldots \vec{v}_n$ lin. dep. then so are $A\vec{v}_1 \ldots A\vec{v}_n$

$\Rightarrow$ $\vec{v}$ is a nontrivial sol'n to $(AB)\vec{x} = \vec{0}$

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

BA is invertible

If AB invertible $\Leftrightarrow$ $Col(B) \cap Null(A) = \{\vec{0}\}$

& $Null(B) = \{\vec{0}\}$

& AB square

$AB\vec{v} = \vec{0} \Rightarrow B\vec{v} \in Null(A)$

$\Rightarrow B\vec{v} = \vec{0}$

$\Rightarrow \vec{v} \in Null(B)$

$\Rightarrow \vec{v} = \vec{0}$

PM #1 1d T/F : For all $A, \vec{b}$ If A is $m \times n$, $\vec{b} \in \mathbb{R}^m$

The set of sol'ns to $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^n$ False

3 checks:

1 Contains $\vec{0}$?

$A \cdot \vec{0} = \vec{0}$ not equal to $\vec{b}$ if $\vec{b} \neq \vec{0}$

So $\vec{0}$ is not a solution to $A\vec{x} = \vec{b}$ if $\vec{b}$ is nonzero.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{0}$ is not in the set of sol'ns of $A\vec{x} = \vec{b}$

A, B $n \times n$, B is invertible, AB is invertible

Show A is invertible.

Because A is square, it's enough to show that $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^n$

Let $\vec{b} \in \mathbb{R}^n$. Since AB invertible, $(AB)\vec{x} = \vec{b}$ has a solution. Let $\vec{v}$ be a solution.

$$A(B\vec{v}) = (AB)\vec{v}$$
$$= \vec{b}$$

$\Rightarrow$ $B\vec{v}$ is a solution to $A\vec{x} = \vec{b}$.

If A, B not square

AB invertible $\not\Rightarrow$ $A\vec{x} = \vec{0}$ has only the trivial sol'n

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, T is invertible if T is 1-to-1 & onto. (T invertible $\Rightarrow n = m$)

If $T: \mathbb{R}^n \to \mathbb{R}^n$, the inverse of T, T^{-1}, is a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$T \circ T^{-1} = id_{\mathbb{R}^n}$$

$$T^{-1} \circ T = id_{\mathbb{R}^n}$$

To calculate, use the standard matrix of T.

$$[T^{-1}]_{std} = [T]_{std}^{-1}$$

To find A^{-1}, $[A \mid I_n] \xrightarrow{\text{row reduce}} [I_n \mid A^{-1}]$

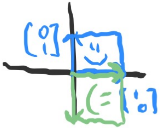
$T : \mathbb{R}^2 \to \mathbb{R}^2$ rotation by 90° counterclockwise

T^{-1} should be rotation by 90° clockwise

$$[T]_{std} = \begin{bmatrix} | & | \\ T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\& \, R_2]{\text{swap } R_1} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 = -R_2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$[T^{-1}]_{std} = [T]_{std}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



$\mathrm{Col}(A)$ = span of the columns of A
$\mathrm{Null}(A)$ = set of solutions to $A\vec{x} = \vec{0}$

$\Big\}$ linear subspaces

basis of a subspace is a set of lin. ind. vectors in the subspace that span the whole subspace

dimension of a subspace = size of any basis
(all bases for a subspace have the same size)

$\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$ = # of pivots when A is put in REF

to find a basis: put A in REF & use columns of A corresponding to pivot columns of REF matrix

$\dim(\mathrm{Null}(A))$ = # of free variables when A is put in REF
to find a basis: write solutions to $A\vec{x} = \vec{0}$ in parametric form

$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix}$

$\dim(\mathrm{Null}(A)) = 1$

$\mathrm{rank}(A) = 2$

A basis for $\mathrm{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix} \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

RREF

$x_1 = -\cancel{x_3}\, t$

$x_2 = -\cancel{x_3}\, t$

x_3 ~~free~~ $= t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

A basis for $\mathrm{Null}(A)$ is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$s \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ A basis for $\mathrm{Null}(A)$ would be $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$