① $A \quad B \quad 2\times 2$ matrices

$$A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix} \qquad B\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-2\\-3\end{bmatrix}$$

Find a solution to $(A+B)\vec{x} = \begin{bmatrix}-1\\2\end{bmatrix}$

Observation: $\begin{bmatrix}1\\5\end{bmatrix} + \begin{bmatrix}-2\\-3\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$

Therefore:

$$\begin{aligned}(A+B)\begin{bmatrix}1\\2\end{bmatrix} &= A\cdot\begin{bmatrix}1\\2\end{bmatrix} + B\begin{bmatrix}1\\2\end{bmatrix}\\ &= \begin{bmatrix}1\\5\end{bmatrix} + \begin{bmatrix}-2\\-3\end{bmatrix}\\ &= \begin{bmatrix}-1\\2\end{bmatrix}\end{aligned}$$

So $\begin{bmatrix}1\\2\end{bmatrix}$ is a solution.

(2) A B 2×2 matrices

$A\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, B is invertible, $B^{-1}=\begin{bmatrix}1&2\\3&4\end{bmatrix}$

Find a nontrivial solution to $(AB)\vec{x}=\vec{0}$

Looks hard because we don't know A.

But, we do know $A\begin{bmatrix}1\\2\end{bmatrix}=\vec{0}$. And for any vector $\vec{v}$, $(AB)\vec{v}=A(B\vec{v})$. So it would be enough to find $\vec{v}\neq\vec{0}$ such that $B\vec{v}=\begin{bmatrix}1\\2\end{bmatrix}$.

Want to solve $B\vec{x}=\begin{bmatrix}1\\2\end{bmatrix}$

$$B\vec{x}=\begin{bmatrix}1\\2\end{bmatrix}\iff \underbrace{B^{-1}(B\vec{x})}_{=\,(B^{-1}B)\vec{x}\,=\,I_2\vec{x}\,=\,\vec{x}}=B^{-1}\begin{bmatrix}1\\2\end{bmatrix}\iff \vec{x}=B^{-1}\begin{bmatrix}1\\2\end{bmatrix}$$

So one solution is $B^{-1}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}5\\11\end{bmatrix}$

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We can check that $B^{-1}\begin{bmatrix}1\\2\end{bmatrix}$ works:

$$(AB)\left(B^{-1}\begin{bmatrix}1\\2\end{bmatrix}\right) = (A(BB^{-1}))\begin{bmatrix}1\\2\end{bmatrix}$$ (matrix multiplication is associative)

$$= (A \cdot I_2)\begin{bmatrix}1\\2\end{bmatrix}$$ (inverses multiply to the identity matrix)

$$= A \cdot \begin{bmatrix}1\\2\end{bmatrix}$$ (any matrix multiplied by the identity matrix is equal to itself)

$$= \begin{bmatrix}0\\0\end{bmatrix}$$ (by assumption)

Moreover, $B^{-1}\begin{bmatrix}1\\2\end{bmatrix}$ is guaranteed to not equal $\begin{bmatrix}0\\0\end{bmatrix}$ because the linear transformation $\vec{v} \longmapsto B^{-1}\cdot\vec{v}$ is 1-to-1

③ What is I_n^{-1}? Answer: I_n

3 ways to explain this:

① The inverse of an invertible $n\times n$ matrix A is the unique $n\times n$ matrix B such that $AB = BA = I_n$

So it's enough to check

$$I_n \cdot I_n = I_n \cdot I_n = I_n$$

which is true because for any $n\times n$ matrix A,

$A \cdot I_n = I_n \cdot A = A$.

② To find the inverse of an invertible $n\times n$ matrix A, you can use row reduction:

$$[A \mid I_n] \xrightarrow{\text{row reduce}} [I_n \mid A^{-1}]$$

For I_n, already in RREF,

$$[I_n \mid I_n] \xrightarrow[\text{needed}]{\text{no row operations}} [I_n \mid I_n]$$

I_n^{-1}

③ For any set X, $id_X^{-1} = id_X$

$$I_n^{-1} = [id_{\mathbb{R}^n}]_{std}^{-1} = [id_{\mathbb{R}^n}^{-1}]_{std} = [id_{\mathbb{R}^n}]_{std} = I_n$$

(4) Find a 2×2 matrix A such that $A \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

One possible answer: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

How to find such an example? Later in the course we will have a more systematic way to do this. For now, here's two approaches.

(1) Need to find $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ such that

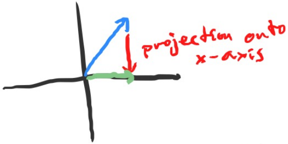
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a^2+bc=0 \qquad ca+dc=0$$
$$ab+bd=0 \qquad cb+d^2=0$$

Try setting $a=0$, $b=0$ to make things easier. Equations above become $dc=0$ $\quad d^2=0 \Rightarrow d=0$

So set $d=0$, $c=1$

② Think about where standard basis vectors $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ get sent. To make sure A is not invertible, try sending $\begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}0\\0\end{bmatrix}$. This corresponds to the linear transformation that projects onto the x-axis



Standard matrix is $\begin{bmatrix}1&0\\0&0\end{bmatrix}$

But there's a problem: vectors on x-axis don't get sent to $\vec{0}$ even when we apply the transformation a second time. One solution is to send $\begin{bmatrix}1\\0\end{bmatrix}$ to $\begin{bmatrix}0\\1\end{bmatrix}$ (i.e. first project onto x-axis and then rotate by $90°$ counterclockwise)

This gives us a linear transformation that sends $\begin{bmatrix}1\\0\end{bmatrix}$ to $\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}0\\0\end{bmatrix}$, so it has standard matrix $\begin{bmatrix}0&0\\1&0\end{bmatrix}$. Squaring it gives $\begin{bmatrix}0&0\\0&0\end{bmatrix}$

⑤ What is the determinant of

$$\begin{bmatrix} 1 & 7 & 8 & 1 & 2 & 3 \\ 2 & -9 & 81 & 2 & 7 & 0 \\ 3 & 4 & 7 & 3 & 7 & -1 \\ 4 & 1 & 1 & 4 & 1 & 1 \\ 5 & 7 & -3 & 5 & 13 & 788 \\ 6 & -1 & -2 & 6 & -4 & -5 \end{bmatrix}$$

This looks hard. But there's a trick. Two of the columns are identical so the columns are linearly dependent.

Cols linearly dependent $\Rightarrow$ free variable in REF

$\Rightarrow$ not invertible

$\Rightarrow$ determinant is 0.

So the answer is 0.