(1a) $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\vec{c} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Is it possible to add together multiples of $\vec{a}$ & $\vec{b}$ to get $\vec{c}$? Yes. $1 \cdot \vec{a} + (-2) \cdot \vec{b} = \vec{c}$

$\Leftrightarrow$ Is $\vec{c}$ a linear combination of $\vec{a}$ and $\vec{b}$?

$\Leftrightarrow$ Is $\vec{c}$ in $\operatorname{span}\{\vec{a}, \vec{b}\}$?

$\Leftrightarrow$ Are there $x_0, x_1 \in \mathbb{R}$ such that

$$x_0 \cdot \vec{a} + x_1 \cdot \vec{b} = \vec{c} \, ?$$

$$x_0 \cdot \vec{a} + x_1 \cdot \vec{b} = x_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_0 + 0 \cdot x_1 \\ 0 \cdot x_0 + 1 \cdot x_1 \end{bmatrix}$$

$\Leftrightarrow$ Are there $x_0, x_1 \in \mathbb{R}$ such that

$$0 \cdot x_0 + 1 \cdot x_1 = 1$$
$$1 \cdot x_0 + 0 \cdot x_1 = -2$$

$\Leftrightarrow$ Is the following system consistent?

columns are $\vec{a}, \vec{b}, \vec{c}$ $\leftarrow$

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & -2 \end{array}\right]$$

Yes because it's in REF with no pivot in the right-hand column

(1b) $\vec{a} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\vec{c} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Is it possible to add together multiples of $\vec{a}$ and $\vec{b}$ to get $\vec{c}$?

This is equivalent to checking if $\left[\begin{array}{cc|c} 3 & 3 & 1 \\ 5 & 2 & -2 \end{array}\right]$ is consistent.

$$\left[\begin{array}{cc|c} 3 & 3 & 1 \\ 5 & 2 & -2 \end{array}\right] \xrightarrow{R_2 = 3R_2 - 5R_1} \left[\begin{array}{cc|c} 3 & 3 & 1 \\ 0 & -9 & -11 \end{array}\right] \quad \text{REF}$$

Yes, it is possible.

Check:

$$\left[\begin{array}{cc|c} 3 & 3 & 1 \\ 0 & -9 & -11 \end{array}\right] \xrightarrow{R_2 = -\frac{1}{9} R_2} \left[\begin{array}{cc|c} 3 & 3 & 1 \\ 0 & 1 & 11/9 \end{array}\right] \xrightarrow{R_1 = R_1 - 3R_2}$$

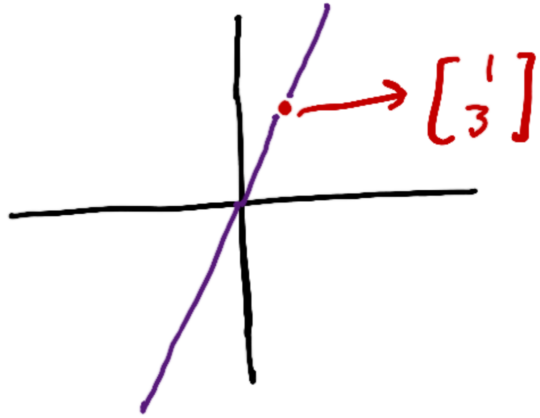
$$\left[\begin{array}{cc|c} 3 & 0 & -8/3 \\ 0 & 1 & 11/9 \end{array}\right] \xrightarrow{R_1 = \frac{1}{3} R_1} \left[\begin{array}{cc|c} 1 & 0 & -8/9 \\ 0 & 1 & 11/9 \end{array}\right] \quad \begin{array}{l} x_0 = -8/9 \\ x_1 = 11/9 \end{array}$$

$$(-8/9)\vec{a} + (11/9)\vec{b} = \begin{bmatrix} -8/3 \\ -40/9 \end{bmatrix} + \begin{bmatrix} 11/3 \\ 22/9 \end{bmatrix} = \begin{bmatrix} 3/3 \\ -18/9 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \vec{c}$$

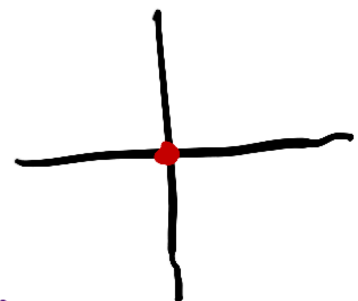
✓

② Draw...

$$\operatorname{span}\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\}$$



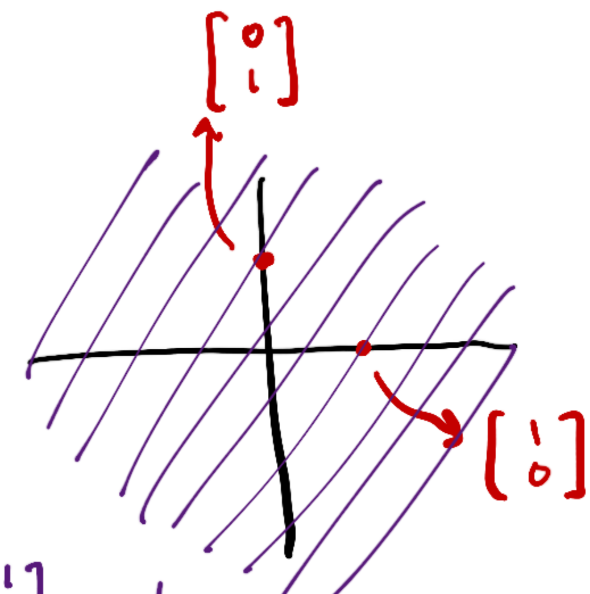
$$\operatorname{span}\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\} = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$$



If you multiply $\begin{bmatrix}0\\0\end{bmatrix}$ by anything, you get $\begin{bmatrix}0\\0\end{bmatrix}$

$$\operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

$$= \mathbb{R}^2$$



You can write any vector in $\mathbb{R}^2$ as a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$

③ $\vec{u}, \vec{v}$ are in span$\{\vec{w}_1, \vec{w}_2\}$

Is $5\vec{u} - 2\vec{v}$ in span$\{\vec{w}_1, \vec{w}_2\}$? Yes.

Proof: Since $\vec{u} \in$ span$\{\vec{w}_1, \vec{w}_2\}$ there are $x_1, x_2 \in \mathbb{R}$ such that $\vec{u} = x_1 \cdot \vec{w}_1 + x_2 \cdot \vec{w}_2$

Since $\vec{v} \in$ span$\{\vec{w}_1, \vec{w}_2\}$ there are $y_1, y_2 \in \mathbb{R}$ such that $\vec{v} = y_1 \cdot \vec{w}_1 + y_2 \cdot \vec{w}_2$

Calculate :

$$5\vec{u} - 2\vec{v} = 5(x_1 \cdot \vec{w}_1 + x_2 \cdot \vec{w}_2) - 2(y_1 \cdot \vec{w}_1 + y_2 \cdot \vec{w}_2)$$
$$= (5x_1 - 2y_1) \cdot \vec{w}_1 + (5x_2 - 2y_2) \cdot \vec{w}_2$$

Since $5\vec{u} - 2\vec{v}$ can be written as a linear combination of $\vec{w}_1$ and $\vec{w}_2$, it is in span$\{\vec{w}_1, \vec{w}_2\}$

④ a) Is it possible to find two vectors in $\mathbb{R}^2$ that don't span all of $\mathbb{R}^2$?

Yes. Example: $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}$

Example 2: $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}$

b) Is it possible to find two vectors in $\mathbb{R}^2$ whose span does not include $\begin{bmatrix}0\\0\end{bmatrix}$?

No. If $\vec{u}, \vec{v} \in \mathbb{R}^2$ then $\begin{bmatrix}0\\0\end{bmatrix} = 0 \cdot \vec{u} + 0 \cdot \vec{v}$ so $\begin{bmatrix}0\\0\end{bmatrix}$ is a linear combination of $\vec{u}$ and $\vec{v}$

c) Is it possible to find two vectors in $\mathbb{R}^3$ whose span is all of $\mathbb{R}^3$?

No. We would need $\begin{bmatrix}a\\b\\c\end{bmatrix}, \begin{bmatrix}d\\e\\f\end{bmatrix}$ such that $\left[\begin{array}{cc|c} a & d & g \\ b & e & h \\ c & f & i \end{array}\right]$ is consistent for all g, h, i which is impossible because there are at most 2 pivots, so not every row has a pivot.