Yesterday's lecture

- Algorithm for finding solutions to systems of linear equations:
  Gaussian elimination/row reduction
- Apply transformations to the system that keep the set of solutions the same
- REF: easy to "read off" # of solutions
- RREF: easy to "read off" set of solutions

1a

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -7 \end{array}\right]$$

$$1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 5$$
$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 = 3$$
$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 = -7$$

$$\Rightarrow \quad x_1 = 5$$
$$x_2 = 3$$
$$x_3 = -7$$

(1b) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -7 \end{array}\right]$

pivots

free variable

$1 \cdot x_1 + 2 \cdot x_2 + 0 \cdot x_3 = 3$

$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 = -7$

Put x_1 and x_3 in terms of x_2

$x_1 = 3 - 2x_2$

$x_3 = -7$

Solutions

$\{ x_1 = 3 - 2t,\ x_2 = t,\ x_3 = -7$ for all $t \in \mathbb{R} \}$

set of all real numbers

1c

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 1 \\ 0 & 2 & 1 & 7 \\ 0 & 0 & 3 & 9 \end{array}\right]$$

REF but not RREF

Can either do more row reduction to get RREF or just use back substitution.

$$\begin{aligned} 1\cdot x_1 + 3\cdot x_2 + 6\cdot x_3 &= 1 \\ 2\cdot x_2 + 1\cdot x_3 &= 7 \\ 3\cdot x_3 &= 9 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_3 &= 3 \\ x_2 &= \frac{7-x_3}{2} = 2 \\ x_1 &= 1-6x_3-3x_2 \\ &= 1-6\cdot 3-3\cdot 2 = -23 \end{aligned}$$

Unique solution:

$$x_1 = -23$$
$$x_2 = 2$$
$$x_3 = 3$$

Check:

$$-23+3\cdot 2+6\cdot 3 = 1$$
$$2\cdot 2+3 = 7 \quad \checkmark$$
$$3\cdot 3 = 9$$

2a

$$3x_1 + 6x_2 + 3x_3 = -3$$
$$5x_1 - 3x_2 + 18x_3 = 8$$
$$7x_1 + 2x_2 + 19x_3 = 5$$

$$\left[\begin{array}{ccc|c} 3 & 6 & 3 & -3 \\ 5 & -3 & 18 & 8 \\ 7 & 2 & 19 & 5 \end{array}\right] \xrightarrow{R_1 = \frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 5 & -3 & 18 & 8 \\ 7 & 2 & 19 & 5 \end{array}\right]$$

$$\xrightarrow{R_2 = R_2 - 5R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & -13 & 13 & 13 \\ 7 & 2 & 19 & 5 \end{array}\right] \xrightarrow{R_3 = R_3 - 7R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & -13 & 13 & 13 \\ 0 & -12 & 12 & 12 \end{array}\right]$$

$$\xrightarrow{R_2 = -\frac{1}{13}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & -12 & 12 & 12 \end{array}\right] \xrightarrow{R_3 = R_3 + 12R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

pivots

free variable

$$\boxed{\begin{array}{l} x_1 = 1 - 3x_3 \\ x_2 = -1 + x_3 \\ x_3 \text{ free} \end{array}}$$

2b

$$x_1 + 2x_2 = 3$$
$$3x_1 - 6x_2 = 9$$
$$x_1 + x_2 = 10$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & -6 & 9 \\ 1 & 1 & 10 \end{array}\right] \xrightarrow{R_2 = R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -12 & 0 \\ 1 & 1 & 10 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_1} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -12 & 0 \\ 0 & -1 & 7 \end{array}\right] \xrightarrow{R_2 = -\frac{1}{12} R_2} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -1 & 7 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + R_2} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{array}\right] \rightarrow \text{inconsistent}$$

No solutions

③ $x_1 + hx_2 = 1$ For what values of h is the system consistent?

$$2x_2 = 2$$

$$3x_1 - x_3 = 3$$

$$\left[\begin{array}{ccc|c} 1 & h & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 3 & 0 & -1 & 3 \end{array}\right] \xrightarrow{R_3 = R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & h & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & -3h & -1 & 0 \end{array}\right] \xrightarrow{R_2 = \frac{1}{2}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & h & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -3h & -1 & 0 \end{array}\right] \xrightarrow{R_3 = R_3 + 3hR_2} \left[\begin{array}{ccc|c} 1 & h & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 3h \end{array}\right] \text{ REF}$$

pivot in every row $\Rightarrow$ system is consistent for all h

Check: $-x_3 = 3h \Rightarrow x_3 = -3h$

$x_2 = 1$

$x_1 = 1 - h \cdot x_2 = 1 - h$

Plug into original equations:

$(1-h) + h = 1$

$2 \cdot 1 = 2$

$3(1-h) - (-3h) = 3$

④ When doing row reduction, why is it not allowed to multiply a row by zero (only by nonzero scalars)?

Multiplying a row by 0 changes the solution set.

Example:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_2 = 0 \cdot R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

unique solution:

$x_1 = 1$

$x_2 = 2$

infinitely many solutions:

$x_1 = 1$

x_2 free

⑤ How many sol'ns if the coefficient matrix in REF has

a) A pivot in every column? 0 or 1

Examples: $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{array}\right]$

Can't have infinitely many because no free variable.

b) A pivot in every row? 1 or infinitely many

Examples: $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right]$

Can't have 0 because no row of all zeros in the coeff. matrix.

c) A free variable? 0 or infinitely many

Examples: $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 2 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$

d) More columns than rows?
0 or infinitely many

Examples: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right]$

Can't have 1 because at most one pivot per row, so some column must be a free variable.

e) More rows than columns: 0, 1, or infinitely many

Examples: $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

(6a) For what values of c is the system consistent?

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & c \end{array}\right]$$ REF

all zeros row in coeff. matrix
system is only consistent if entry in the column on the right is also 0.

$\boxed{\text{Consistent if } c = 0}$

(6b)

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ c & 3 & -2 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - cR_1} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 3-2c & -2-3c \\ 0 & 0 & 0 \end{array}\right] \quad \text{REF}$$

if $3-2c \neq 0$ then it is a pivot and the system is consistent. Otherwise this is a row of all zeros in the coeff. matrix & the system is only consistent if $-2-3c = 0$. But if $c = 3/2$ so $3-2c = 0$ then $-2-3c \neq 0$.

Consistent if $3-2c \neq 0$, i.e. if $c \neq 3/2$.