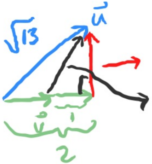
Review Suppose $\vec{u}, \vec{v} \in \mathbb{R}^5$ $\quad \vec{u}\cdot\vec{u} = 13 \quad \vec{u}\cdot\vec{v} = 2$
and $\vec{v}$ is a unit vector.
What is the distance between $\vec{u}$ and $\vec{v}$?



$$\sqrt{\sqrt{13}^2 - 2^2} = \sqrt{13-4} = \sqrt{9} = 3$$

$$\sqrt{3^2+1^2} = \sqrt{9+1} = \sqrt{10}$$

distance between $\vec{u}$ and $\vec{v}$: $\|\vec{u}-\vec{v}\|$

$$= \sqrt{(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})} = \sqrt{10}$$

$$(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$
$$= 13 - 2 - 2 + 1$$
$$= 10$$

Fourier Series

In $\mathbb{R}^2$ if $\vec{v}_1, \vec{v}_2$ are an orth. basis then

$$\vec{u} = \frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{u}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

Idea: In $C[-\pi,\pi]$ with inner product

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$$

$1, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \ldots$

are orthogonal

↳ better to use $e^{0ix}, e^{ix}, e^{2ix}, e^{3ix}, \ldots$

Not a basis.

But if "infinite linear combinations" are allowed then they act like a basis

For any "nice enough" $f \in C[-\pi,\pi]$

$$f(x) = \frac{\langle f,1\rangle}{\underbrace{\langle 1,1\rangle}_{=2\pi}}1 + \frac{\langle f,\sin(x)\rangle}{\underbrace{\langle \sin(x),\sin(x)\rangle}_{=\pi}}\sin(x) + \frac{\langle f,\cos(x)\rangle}{\underbrace{\langle \cos(x),\cos(x)\rangle}_{=\pi}}\cos(x) + \ldots$$

↳ Fourier series of f

① Calculate the fourier series of $f(x)=x$

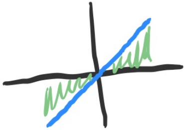
$$\langle f, 1\rangle = \int_{-\pi}^{\pi} f(x)\,dx = 0 \qquad \langle 1,1\rangle = 2\pi$$

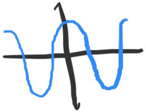
$$\langle f, \cos(nx)\rangle = \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0 \qquad \langle \cos(nx), \cos(nx)\rangle = \pi$$

$$\langle \sin(nx), \sin(nx)\rangle = \pi$$

$$\langle f, \sin(nx)\rangle = \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \int_{-\pi}^{\pi} x\sin(nx)\,dx$$





$u = x \qquad dv = \sin(nx)\,dx$

$du = dx \qquad v = -\frac{1}{n}\cos(nx)$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\frac{2\pi}{n}}{\pi}\sin(nx)$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin(nx)$$

$$= -\frac{x}{n}\cos(nx)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -\frac{1}{n}\cos(nx)\,dx$$

$$= -\frac{x}{n}\cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n^2}\sin(nx)\Big|_{-\pi}^{\pi}$$

$= 0$

$$= -\frac{\pi}{n}\cos(n\pi) - \left(-\frac{(-\pi)}{n}\cos(n(-\pi))\right) + \frac{1}{n^2}\sin(n\pi) - \frac{1}{n^2}\sin(n(-\pi))$$

$$= -\frac{2\pi}{n}\cos(n\pi) = \begin{cases} -\frac{2\pi}{n} & \text{if } n \text{ even} \\ \frac{2\pi}{n} & \text{if } n \text{ odd} \end{cases}$$