Review Suppose $\vec{u} \in \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{=\vec{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{=\vec{v}_2} \right\}$

$$\vec{u} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 5 \qquad \vec{u} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -3 \qquad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

Find $\vec{u}$.

Since $\vec{u} \in \text{span}\{\vec{v}_1, \vec{v}_2\}$ there are $a, b \in \mathbb{R}$ s.t.

$$\vec{u} = a\vec{v}_1 + b\vec{v}_2 = \begin{bmatrix} a \\ b \\ a \end{bmatrix} \qquad \begin{bmatrix} a \\ b \\ a \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2a \qquad \begin{bmatrix} a \\ b \\ a \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = b$$

$$\vec{u} \cdot \vec{v}_1 = (a\vec{v}_1 + b\vec{v}_2) \cdot \vec{v}_1 = a\vec{v}_1 \cdot \vec{v}_1 + b\cancelto{0}{\vec{v}_2 \cdot \vec{v}_1} = a\vec{v}_1 \cdot \vec{v}_1$$

$$\vec{u} \cdot \vec{v}_2 = (a\vec{v}_1 + b\vec{v}_2) \cdot \vec{v}_2 = a\cancelto{0}{\vec{v}_1 \cdot \vec{v}_2} + b\vec{v}_2 \cdot \vec{v}_2 = b\vec{v}_2 \cdot \vec{v}_2$$

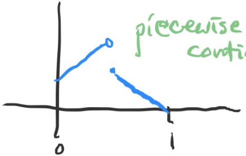
$$a = \frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \frac{5}{1^2 + 0^2 + 1^2} = \frac{5}{2} \qquad b = \frac{\vec{u} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} = \frac{-3}{1} = -3$$

$$\vec{u} = \frac{5}{2}\vec{v}_1 - 3\vec{v}_2 = \begin{bmatrix} 5/2 \\ -3 \\ 5/2 \end{bmatrix}$$

Inner Product Spaces

$\mathbb{R}^n$ $\longrightarrow$ Abstract vector spaces

dot product $\longrightarrow$ Inner product



- An operation on a vector space V
- Takes two vectors from V & gives a scalar
- Satisfies some list of properties which contain all the basic facts about the dot product

Things like projections, Gram-Schmidt, etc work in any inner product space

$\hookrightarrow$ vector space with an inner product

Example: $C[-\pi,\pi]$ = continuous functions $[-\pi,\pi]\to\mathbb{R}$

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$$

① Find the norm of $\sin(x)$ in the inner product space $C[-\pi,\pi]$ with inner product

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$$

def: $\|\vec{v}\| = \sqrt{\langle \vec{v},\vec{v}\rangle}$

Hint: $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$ $\qquad \sin(x) = \dfrac{e^{ix}-e^{-ix}}{2i}$

In $\mathbb{R}^n$, $\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$ $\qquad (\sin(x))^2 = \dfrac{(e^{ix}-e^{-ix})^2}{(2i)^2} = \ldots$

$$\|\sin(x)\| = \sqrt{\langle \sin(x),\sin(x)\rangle} = \sqrt{\int_{-\pi}^{\pi}\sin(x)\sin(x)\,dx} = \sqrt{\pi}$$

$$\int_{-\pi}^{\pi}\sin^2(x)\,dx = \int_{-\pi}^{\pi}\frac{1-\cos(2x)}{2}\,dx = \frac{1}{2}\left(\int_{-\pi}^{\pi}dx - \int_{-\pi}^{\pi}\cos(2x)\,dx\right)$$

$u = 2x$

$du = 2dx \qquad dx = \dfrac{du}{2}$

$$= \frac{1}{2}\left(2\pi - \int_{-2\pi}^{2\pi}\cos(u)\frac{du}{2}\right)$$

$$= \frac{1}{2}\left(2\pi - \frac{1}{2}\left(\sin(u)\Big|_{-2\pi}^{2\pi}\right)\right) = \frac{1}{2}\left(2\pi - \frac{1}{2}\cdot 0\right) = \pi$$

② Find the projection of $\sin(x)$ onto $\operatorname{span}\{1, x\}$ in the inner product space $C[-\pi, \pi]$ with inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$$

In $\mathbb{R}^n$, $\operatorname{proj}_{\operatorname{span}\{\vec{v}_1, \vec{v}_2\}}(\vec{u}) = \dfrac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \dfrac{\vec{u}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$

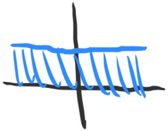
This only works when $\vec{v}_1$ & $\vec{v}_2$ are orthogonal

$$\langle 1, x\rangle = \int_{-\pi}^{\pi} 1\cdot x\,dx = \frac{x^2}{2}\Big|_{-\pi}^{\pi} = \frac{\pi^2}{2} - \frac{(-\pi)^2}{2} = 0$$

$$\operatorname{proj}_{\operatorname{span}\{1,x\}}(\sin(x)) = \frac{\langle \sin(x), 1\rangle}{\langle 1, 1\rangle}1 + \frac{\langle \sin(x), x\rangle}{\langle x, x\rangle}x$$

$$= \frac{\int_{-\pi}^{\pi}\sin(x)\,dx}{\int_{-\pi}^{\pi} 1\,dx}1 + \frac{\int_{-\pi}^{\pi} x\sin(x)\,dx}{\int_{-\pi}^{\pi} x^2\,dx}x$$

" 2π



$x\Big|_{-\pi}^{\pi}$

$= \pi - (-\pi)$

$$\int_{-\pi}^{\pi} \sin(x)\,dx = -\cos(x)\Big|_{-\pi}^{\pi} = -(-1) - (-(-1)) = 0$$

$$\int_{-\pi}^{\pi} x\sin(x)\,dx = -x\cos(x)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos(x)\,dx = -x\cos(x)\Big|_{-\pi}^{\pi}$$

$u = x \quad dv = \sin(x)dx$

$du = 1dx \quad v = -\cos(x)$

$$= -\pi(-1) - (-(-\pi)(-1))$$

$$= \pi + \pi = 2\pi$$

$$\int_{-\pi}^{\pi} x^2\,dx = \frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{\pi^3}{3} - \frac{(-\pi)^3}{3} = \frac{2\pi^3}{3}$$

$$\text{proj}_{\text{span}\{1,x\}}(\sin(x)) = \frac{0}{?}1 + \frac{2\pi}{(2\pi^3/3)}x$$

$$= \boxed{\frac{3}{\pi^2}x}$$