Review Find a homogeneous system of linear 1^{st} order ODEs s.t.

$$\left.\begin{aligned} y_1(t) &= 2e^{3t} + e^{-2t} \\ y_2(t) &= e^{3t} + 2e^{-2t} \end{aligned}\right\} \quad \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = e^{3t}\underline{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + e^{-2t}\underline{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$v'(t) = Av(t)$$

is a sol'n.

Find A s.t. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvec. w/ eigenval 3

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvec w/ eigenval -2

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \qquad v'(t) = Av(t)$$

$$\text{std} \leftarrow \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} \qquad \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} \leftarrow \text{std}$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\left(\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 6 & -2 \\ 3 & -4 \end{bmatrix}\left(\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 14 & -10 \\ 10 & -11 \end{bmatrix}$$

$$y_1'(t) = \frac{14}{3}y_1(t) - \frac{10}{3}y_2(t)$$

$$y_2'(t) = \frac{10}{3}y_1(t) - \frac{11}{3}y_2(t)$$

① Find a solution to

$$y'(t) = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} y(t)$$

$$y(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

General sol'n: $\underline{c_1} e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \underline{c_2} e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$c_1 e^{2 \cdot 0} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3 \cdot 0} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 5 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} c_1 = 2 \\ c_2 = 1 \end{array}$$

Sol'n: $2e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Complex Eigenvalues

A is 2×2 real matrix

$$y'(t) = Ay(t)$$

A has eigenvalues : $a \pm bi \quad a,b \in \mathbb{R}$

$a+bi$ has eigenvector : $\vec{u} + i\vec{v} \quad \vec{u}, \vec{v} \in \mathbb{R}^2$ ($\Rightarrow$ $\vec{u} - i\vec{v}$ eigenvec for $a-bi$)

general sol'n : $c_1 e^{(a+bi)t}(\vec{u} + i\vec{v}) + c_2 e^{(a-bi)t}(\vec{u} - \vec{v}i)$

$$e^{(a+bi)t}(\vec{u} + i\vec{v}) = e^{at}(\cos(bt) + i\sin(bt))(\vec{u} + i\vec{v})$$

$$= e^{at}\cos(bt)\vec{u} + i\, e^{at}\sin(bt)\vec{u} + i\, e^{at}\cos(bt)\vec{v} - e^{at}\sin(bt)\vec{v}$$

$$= \underbrace{\left(e^{at}\cos(bt)\vec{u} - e^{at}\sin(bt)\vec{v}\right)}_{\text{is a sol'n}} + i\underbrace{\left(e^{at}\sin(bt)\vec{u} + e^{at}\cos(bt)\vec{v}\right)}_{\text{is a sol'n}}$$

alt. general sol'n :

$$c_1\left(e^{at}\cos(bt)\vec{u} - e^{at}\sin(bt)\vec{v}\right) + c_2\left(e^{at}\sin(bt)\vec{u} + e^{at}\cos(bt)\vec{v}\right)$$

② Find the general sol'n to

$$y'(t) = \begin{bmatrix} 5 & 1 \\ -8 & 1 \end{bmatrix} y(t)$$

"A"

① $$\chi_A(t) = \det\begin{bmatrix} 5-t & 1 \\ -8 & 1-t \end{bmatrix} = (5-t)(1-t)+8$$

$$= 5 - t - 5t + t^2 + 8$$

$$= t^2 - 6t + 13$$

roots: $$\frac{6 \pm \sqrt{36-4\cdot 13}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$$

② eigenvector for $3+2i$:

$$\begin{bmatrix} 5-(3+2i) & 1 \\ -8 & 1-(3+2i) \end{bmatrix} = \begin{bmatrix} 2-2i & 1 \\ -8 & -2-2i \end{bmatrix}$$ not invertible

$(2-2i)x_1 + x_2 = 0$
choose $x_1 = 1$ $\Rightarrow$ $x_2 = -2+2i$ $\begin{bmatrix} 1 \\ -2+2i \end{bmatrix}$

③ General sol'n: $$c_1\left(e^{3t}\cos(2t)\begin{bmatrix} 1 \\ -2 \end{bmatrix} - e^{3t}\sin(2t)\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$$
$$+ c_2\left(e^{3t}\sin(2t)\begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{3t}\cos(2t)\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$$

Reducing 2^{nd} order ODEs to systems of 1^{st} order ODEs

$ay'' + by' + cy = 0$ char. poly. $at^2 + bt + c = 0$

Introduce new variable $x(t) = y'(t)$

$$\begin{cases} y'(t) = x(t) \\ x'(t) = y''(t) = -\frac{b}{a}y'(t) - \frac{c}{a}y(t) \end{cases}$$

$$\begin{bmatrix} y'(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \begin{bmatrix} y(t) \\ x(t) \end{bmatrix}$$

③ Calculate the char. poly. of

$$\det\begin{bmatrix} -t & 1 \\ -\frac{c}{a} & -\frac{b}{a} - t \end{bmatrix} = -t\left(-\frac{b}{a} - t\right) + \frac{c}{a}$$

$$= t^2 + \frac{b}{a}t + \frac{c}{a}$$