Solving systems of 1^{st} order linear ODEs

$$x'(t) = 5x(t) - 2y(t)$$
$$y'(t) = 3x(t) + 3y(t)$$

$$\rightsquigarrow \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Can think of this as a "vector-valued" ODE

$\mathbb{R} \to \mathbb{R}^2$

$$v'(t) = \begin{bmatrix} 5 & -2 \\ 3 & 3 \end{bmatrix} v(t)$$

take derivative of each component

function $\mathbb{R} \to \mathbb{R}^2$

① Check if each function is a sol'n to

$$y'(t) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} y(t)$$

a) $y_1(t) = \begin{bmatrix} -e^{3t} \\ e^{3t} \\ 4e^{3t} \end{bmatrix}$ $\quad y_1'(t) = \begin{bmatrix} -3e^{3t} \\ 3e^{3t} \\ 12e^{3t} \end{bmatrix}$ Is a solution

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \begin{bmatrix} -e^{3t} \\ e^{3t} \\ 4e^{3t} \end{bmatrix} = \begin{bmatrix} -e^{3t} + 2e^{3t} - 4e^{3t} \\ -e^{3t} + 4e^{3t} \\ -4e^{3t} - 4e^{3t} + 20e^{3t} \end{bmatrix} = \begin{bmatrix} -3e^{3t} \\ 3e^{3t} \\ 12e^{3t} \end{bmatrix}$$

b) $y_2(t) = \begin{bmatrix} \sin(t) \\ 2 \\ 3e^{5t} \end{bmatrix}$ $\quad y_2'(t) = \begin{bmatrix} \cos(t) \\ 0 \\ 15e^{5t} \end{bmatrix}$ Not a solution

$\neq$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \begin{bmatrix} \sin(t) \\ 2 \\ 3e^{5t} \end{bmatrix} = \begin{bmatrix} \sin(t) + 4 - 3e^{5t} \\ \sin(t) + 3e^{5t} \\ 4\sin(t) - 8 + 15e^{5t} \end{bmatrix}$$

How to solve systems of ODEs $\quad y'(t) = Ay(t)$

Key idea: eigenvectors of A give solutions

(2) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A

with eigenval. 5

Show $\begin{bmatrix} e^{5t} \\ 2e^{5t} \end{bmatrix}$ is a sol'n to $y'(t) = Ay(t)$

derivative: $\begin{bmatrix} 5e^{5t} \\ 10e^{5t} \end{bmatrix} = 5e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

multiply by A: $A\begin{bmatrix} e^{5t} \\ 2e^{5t} \end{bmatrix} = A\left(e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

$$= e^{5t}\left(A\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$= e^{5t}\left(5\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$= 5e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Algorithm for solving systems of ODEs $\quad y'(t) = Ay(t)$

① Find eigenvectors & eigenvalues of A

② For each eigenvec. $\vec{v}$ with eigenval. λ,

$e^{\lambda t}\vec{v}$ is a solution

③ If A is diagonalizable, every sol'n is a lin. comb. of solutions coming from eigenvectors

If A is not diagonalizable...

If A is 2×2 with eigenvals λ_1, λ_2

eigenvecs $\vec{v}_1, \vec{v}_2$

general sol'n: $c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

③ Find the general sol'n to

$$y'(t) = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}}_{A} y(t)$$

$$\chi_A(t) = \det \begin{bmatrix} 1-t & 2 \\ 2 & -2-t \end{bmatrix} = (1-t)(-2-t) - 2\cdot 2$$

$$= -2 + t + t^2 - 4$$

$$= t^2 + t - 6 = (t-2)(t+3)$$

roots: $2, -3$

E_2 : $\begin{bmatrix} 1-2 & 2 \\ 2 & -2-2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$ $\quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

E_{-3} : $\begin{bmatrix} 1+3 & 2 \\ 2 & -2+3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

general sol'n: $c_1 \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-3t} \\ -2e^{-3t} \end{bmatrix}$

$$= c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$