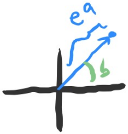
Complex Numbers

① $e^{a+bi} = e^{a}(\cos(b) + i\sin(b))$



$$\Rightarrow \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

② Suppose $ay'' + by' + cy = 0$ has a complex root for its char. polynomial $a, b, c \in \mathbb{R}$

$x + iy$ is a root

- Complex roots come in conjugate pairs

  $x + iy$, $x - iy$ are both roots

- $c_1 e^{(x+iy)t} + c_2 e^{(x-iy)t}$ is still a solution

  $\Rightarrow$ $e^{xt}\cos(yt)$ and $e^{xt}\sin(yt)$ are both also sol'ns

Algorithm for solving homogeneous linear ODE:

1. Write char. polynomial
2. Find roots
3. For each real root a, e^{at} is a sol'n
4. If a is repeated n times,

   $e^{at}, te^{at}, t^2e^{at}, \ldots, t^{(n-1)}e^{at}$ sol'ns

5. If $a+bi$ is a root then so is $a-bi$
   $(b \neq 0)$

   and $e^{at}\cos(bt)$, $e^{at}\sin(bt)$ are sol'ns

General sol'n: arbitrary linear combination of the sol'ns above

1. Find the general solution for each ODE

a) $y'' - 6y' + 10y = 0$

$\lambda^2 - 6\lambda + 10$

roots: $\dfrac{6 \pm \sqrt{36-40}}{2} = 3 \pm \dfrac{\sqrt{-4}}{2}$

$= 3 \pm i$

General sol'n: $C_1 e^{3t}\cos(t) + C_2 e^{3t}\sin(t)$

b) $y'' + 4y' + 6y = 0$

$\lambda^2 + 4\lambda + 6$

roots: $\dfrac{-4 \pm \sqrt{16 - 4\cdot 6}}{2}$

$= \dfrac{-4 \pm \sqrt{-8}}{2}$

$= -2 \pm \sqrt{2}\, i$

General sol'n: $C_1 e^{-2t}\cos(\sqrt{2}\,t) + C_2 e^{-2t}\sin(\sqrt{2}\,t)$

c) $y^{(4)} + 8y'' + 16y = 0$

$\lambda^4 + 8\lambda^2 + 16 = 0$

$\lambda^2 = \dfrac{-8 \pm \sqrt{64 - 4\cdot 16}}{2}$

$= -4$

$\Rightarrow \lambda = \pm\sqrt{-4}$

$= \pm 2i$

$(\lambda^2 + 4)^2 = (\lambda + 2i)^2(\lambda - 2i)^2$

General sol'n:
$C_1 \cos(2t) + C_2 \sin(2t) + C_3 t\cos(2t) + C_4 t\sin(2t)$

Check:

$(e^{3t}\cos(t))'' - 6(e^{3t}\cos(t))' + 10e^{3t}\cos(t)$

$= 10e^{3t}\cancel{\cos(t)} - 6(\cancel{3e^{3t}}\cos(t) - \cancel{e^{3t}}\sin(t))$

$+ (\cancel{9e^{3t}}\cos(t) - \cancel{3e^{3t}}\sin(t) - \cancel{3e^{3t}\sin}(t) - \cancel{e^{3t}\cos(t)})$

Initial value problems

$$ay'' + by' + cy = 0 \qquad \text{vs.} \qquad ay'' + by' + cy = 0$$

$$y(5) = 10 \qquad\qquad y(5) = 10$$

$$y'(5) = 13 \qquad\qquad y'(6) = 13$$

a solution always exists & is unique

sol'n not guaranteed to exist or to be unique

Algorithm

① General sol'n: $c_1 y_1(t) + c_2 y_2(t)$

② $c_1 y_1(5) + c_2 y_2(5) = 10$

$c_1 y_1'(5) + c_2 y_2'(5) = 13$

} solve for c_1, c_2 using linear algebra

② a) $y'' + y' = 0$

$y(0) = 2$

$y'(0) = 1$

$\lambda^2 + \lambda = \lambda(\lambda + 1)$

roots: $0, -1$

general sol'n: $c_1 e^{0t} + c_2 e^{-t} = c_1 + c_2 e^{-t}$

$c_1 + c_2 e^{-0} = 2 \Rightarrow c_1 - 1 = 2 \Rightarrow c_1 = 3$

$-c_2 e^{-0} = 1 \Rightarrow c_2 = -1$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

$\boxed{3 - e^{-t}}$

b) $y''' + 5y'' + 4y' = 0$

$y(0) = 8$

$y'(0) = -9$

$y''(0) = 33$

$\lambda^3 + 5\lambda^2 + 4\lambda = \lambda(\lambda^2 + 5\lambda + 4)$

$= \lambda(\lambda + 4)(\lambda + 1)$

roots: $0, -4, -1$

general sol'n: $c_1 + c_2 e^{-4t} + c_3 e^{-t}$

$\boxed{5 + 2e^{-4t} + e^{-t}}$

$c_1 + c_2 + c_3 = 8$

$-4c_2 - c_3 = -9$

$16c_2 + c_3 = 33$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & -4 & -1 & -9 \\ 0 & 16 & 1 & 33 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & -4 & -1 & -9 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

$c_1 = 5$

$c_2 = 2$

$c_3 = 1$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & -4 & -1 & -9 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & -4 & 0 & -8 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

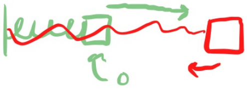
③ show that

$$\left.\begin{aligned} y'' + y &= 0 \\ y(0) &= 0 \\ y'(\pi/2) &= 1 \end{aligned}\right\}$$ motion of a spring

period of spring is independent of the initial pos. & velocity

has no solution.

$\pi/2 \to$ 1/4 the period



$\lambda^2 + 1$

roots : $\pm i$

general solution: $c_1 \cos(t) + c_2 \sin(t)$

$$c_1 \cos(0) + c_2 \sin(0) = 0 \rightsquigarrow c_1 = 0$$

$$-c_1 \sin(\pi/2) + c_2 \cos(\pi/2) = 1 \rightsquigarrow -c_1 = 1$$