Reminder

Spectral Thm: If A is a real, $n\times n$ symmetric matrix

$\Rightarrow$
- $\mathbb{R}^n$ has orthonormal basis of eigenvecs of A
- All eigenvals of A are real

Application: If A is <u>any</u> real $n\times m$ matrix

- A^TA is symmetric & square
- A^TA is orthog. diagonalizable
- All eigenvals of A^TA are nonnegative real #s

rank(A) = # of nonzero singular values

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$ eigenvals of A^TA

right singular vectors of A

$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ eigenvecs of A^TA

$A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_m$ $\leftarrow$ orthogonal & span Col(A)

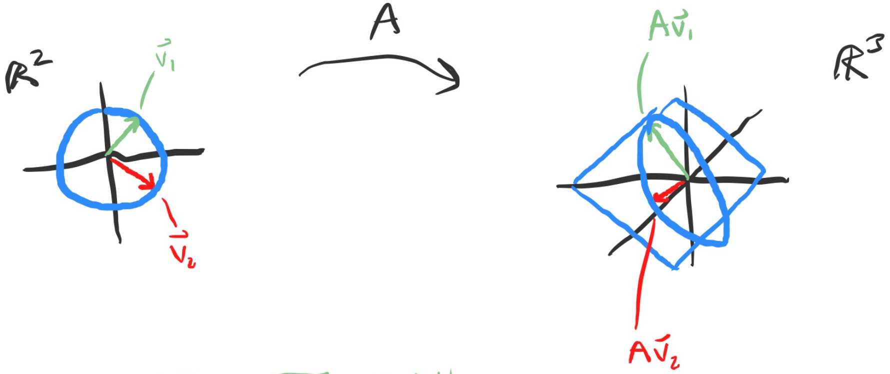
left singular vectors of A

$\sigma_1 = \sqrt{\lambda_1} \geq \sigma_2 = \sqrt{\lambda_2} \geq \dots \geq \sigma_m = \sqrt{\lambda_m}$

singular values of A

Geometric Picture:

$$A = \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$$



$$\|A\vec{v}_1\| = \sqrt{\sigma_1} \cdot \|\vec{v}_1\|$$

$$\|A\vec{v}_2\| = \sqrt{\sigma_2} \cdot \|\vec{v}_2\|$$

(i) Orthogonally diagonalize A^TA where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \quad A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\begin{aligned} \chi_{A^TA}(t) = \det \begin{bmatrix} 3-t & -1 \\ -1 & 3-t \end{bmatrix} &= (3-t)^2 - (-1)^2 \\ &= 9 - 6t + t^2 - 1 \\ &= t^2 - 6t + 8 \\ &= (t-4)(t-2) \end{aligned}$$

Eigenvalues of A^TA: 4, 2

Eigenspace for 4: $E_4 = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$

$$A^TA - 4I_2 = \begin{bmatrix} 3-4 & -1 \\ -1 & 3-4 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = -x_2 \\ x_2 \text{ free} \end{array}$$

$$\left\| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

Eigenspace for 2: $E_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$

$$A^TA - 2I_2 = \begin{bmatrix} 3-2 & -1 \\ -1 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = x_2 \\ x_2 \text{ free} \end{array}$$

$$\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$A^TA = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

② Find $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

and compute their norms

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -2/\sqrt{2} \\ 0 \\ -2/\sqrt{2} \end{bmatrix}$$

$$\text{norm} = \sqrt{(-2/\sqrt{2})^2 + 0^2 + (-2/\sqrt{2})^2} = \sqrt{2+2} = \sqrt{4} = 2$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 2/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\text{norm} = \sqrt{0^2 + (2/\sqrt{2})^2 + 0^2} = \frac{2}{\sqrt{2}} = \frac{2\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}$$

square roots of corresponding eigenvalues of A^TA

Application: PCA (principal components analysis)

Karhunen-Loève transform (signal processing)

Hotelling transform

Proper orthog. decomposition (mech. eng.)

Eckart-Young theorem

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Collect lots of data, much of it probably irrelevant

Goal: Boil it down to the important parts

Example: data = pictures of faces

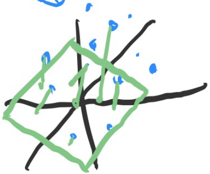
picture $\rightsquigarrow$ $\begin{bmatrix} 0.5 \\ 1 \\ 2.3 \\ \vdots \\ 18 \end{bmatrix}$ → values of each pixel

= vector in some high dimensional space

"Curse of dimensionality"

Goal: Find a low dimensional subspace so that if you project each data point onto it you still retain most of the information

data points



$$X = \begin{bmatrix} \text{---} & x_1 & \text{---} \\ & \vdots & \\ \text{---} & x_m & \text{---} \end{bmatrix}$$

← each row is a data point

Best K-dim. subspace is the span of the 1st K singular vectors of X