Announcements

① 1-on-1 meeting times available for today on bcourses

1-2 pm PDT

8-10 pm PDT

If you want to have a 1-on-1 meeting after today, send me an email

② If you want a copy of the recording of my first discussion section today, send me an email

Review Find a vector which is orthogonal to both

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix}$$

Want $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that

$$0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = a \cdot 1 + b \cdot 2 + c \cdot (-3)$$

$$0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix} = a \cdot 0 + b \cdot (-6) + c \cdot 4$$

Check:

$$\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = 5 + 4 - 9 = 0$$

$$\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix} = 0 - 12 + 12 = 0$$

$\boxed{\text{If you want to find a vector orthogonal to } \vec{v}_1, \dots, \vec{v}_n \text{ just take any vector in } \mathrm{Null}\left(\begin{bmatrix} - v_1 - \\ \vdots \\ - v_n - \end{bmatrix}\right)}$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -6 & 4 & 0 \end{array}\right] \xrightarrow{R_2 = -\frac{1}{6} R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -2/3 & 0 \end{array}\right] \xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5/3 & 0 \\ 0 & 1 & -2/3 & 0 \end{array}\right]$$

$x_1 = (5/3) x_3$

$x_2 = (2/3) x_3$

x_3 free

$$\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$$ (or any scalar multiple of it)

Orthogonal Bases

Suppose $\vec{v}_1, \ldots, \vec{v}_n$ are orthogonal and nonzero

① $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent

② If $\vec{u} \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$

$$\vec{u} = \frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \ldots + \frac{\vec{u}\cdot\vec{v}_n}{\vec{v}_n\cdot\vec{v}_n}\vec{v}_n$$

Why does this work?

There are scalars $a_1 \ldots a_n$ s.t. $\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n$

$$\begin{aligned}
\vec{u}\cdot\vec{v}_1 &= (a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n)\cdot\vec{v}_1 \\
&= a_1\vec{v}_1\cdot\vec{v}_1 + a_2\underbrace{\vec{v}_2\cdot\vec{v}_1}_{=0} + \ldots + a_n\underbrace{\vec{v}_n\cdot\vec{v}_1}_{=0} \\
&= a_1\vec{v}_1\cdot\vec{v}_1
\end{aligned}$$

$$a_1 = \frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}$$

(1) $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$

$W = \text{span}\{\vec{v}_1, \vec{v}_2\}$

$B = \{\vec{v}_1, \vec{v}_2\}$ basis for W

What is $[\vec{u}]_B$?

$\vec{u}\cdot\vec{v}_1 = 1\cdot 1 + 0\cdot 1 + 5\cdot 1 = 6$

$\vec{u}\cdot\vec{v}_2 = 1\cdot 1 + 0\cdot 2 + 5\cdot(-3) = -14$

$\vec{v}_1\cdot\vec{v}_1 = 1^2 + 1^2 + 1^2 = 3$

$\vec{v}_2\cdot\vec{v}_2 = 1^2 + 2^2 + (-3)^2 = 14$

$$\vec{u} = \frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{u}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

$$= \frac{6}{3}\vec{v}_1 + \frac{-14}{14}\vec{v}_2$$

$$= 2\vec{v}_1 - \vec{v}_2$$

$$[\vec{u}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Check:

$$2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2-1 \\ 2-2 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \checkmark$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & -3 & 5 \end{array}\right] \rightsquigarrow$$

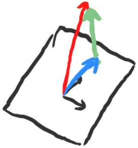
Orthogonal Projections and Best Approximation

If $\vec{v}_1 \ldots \vec{v}_n$ are orthogonal and nonzero and $\vec{u} \notin \text{span}\{\vec{v}_1, \ldots \vec{v}_n\}$

then

$$\text{proj}_{\text{span}\{\vec{v}_1 \ldots \vec{v}_n\}} = \hat{u} = \frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \ldots + \frac{\vec{u} \cdot \vec{v}_n}{\vec{v}_n \cdot \vec{v}_n} \vec{v}_n$$

is the vector in $\text{span}\{\vec{v}_1, \ldots \vec{v}_n\}$ that is closest to $\vec{u}$



More generally, if W is any subspace of $\mathbb{R}^n$, we can find $\text{proj}_W(\vec{u})$ by:

1. Find an orthogonal basis $\vec{v}_1, \ldots, \vec{v}_K$ for W
2. $\text{proj}_W(\vec{u}) = \hat{u} = \frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \ldots + \frac{\vec{u} \cdot \vec{v}_K}{\vec{v}_K \cdot \vec{v}_K} \vec{v}_K$

Distance from $\vec{u}$ to W is the distance from $\vec{u}$ to $\text{proj}_W(\vec{u})$

Decomposition of $\vec{u}$ into a vector in W and a vector orthogonal to everything in W:

$$\vec{u} = \text{proj}_W(\vec{u}) + \underbrace{(\vec{u} - \text{proj}_W(\vec{u}))}_{\text{orthogonal to } W}$$

① $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $W = \text{span}\{\vec{v}_1, \vec{v}_2\}$

a) $\text{proj}_W(\vec{y}) = \dfrac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \dfrac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 = \dfrac{2}{3}\vec{v}_1 + \dfrac{1}{2}\vec{v}_2$

$\vec{y}\cdot\vec{v}_1 = 1+1+0 = 2$

$\vec{y}\cdot\vec{v}_2 = 1+0+0 = 1$

$\vec{v}_1\cdot\vec{v}_1 = 1^2+1^2+1^2 = 3$

$\vec{v}_2\cdot\vec{v}_2 = 1^2+0^2+(-1)^2 = 2$

$= \begin{bmatrix} 2/3 + 1/2 \\ 2/3 + 0 \\ 2/3 - 1/2 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 2/3 \\ 1/6 \end{bmatrix}$

b) Distance from $\vec{y}$ to W = $\text{dist}(\vec{y}, \text{proj}_W(\vec{y}))$

$= \sqrt{(1-7/6)^2 + (1-2/3)^2 + (0-1/6)^2} = \sqrt{(-1/6)^2 + (1/3)^2 + (-1/6)^2}$

$= \sqrt{1/36 + 4/36 + 1/36} = \sqrt{6/36} = \sqrt{1/6}$

c) Find $\hat{y}$ and z s.t. $\hat{y} \in W$, $\vec{z} \perp W$ and $\vec{y} = \hat{y} + \vec{z}$

$\hat{y} = \text{proj}_W(\vec{y}) = \begin{bmatrix} 7/6 \\ 1/3 \\ 1/6 \end{bmatrix}$

$\vec{y} - \text{proj}_W(\vec{y})$ is orthogonal to everything in W

$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} -1/6 \\ 1/3 \\ -1/6 \end{bmatrix}$

Orthogonal Bases

(2) Solve the system of linear equations without doing row reduction (hint: the columns of the coefficient matrix are orthogonal to each other):

$$
\begin{aligned}
x_1 + 6x_2 + 2x_3 &= 23 \\
2x_1 - x_2 + x_3 &= 1 \\
3x_1 \qquad - 16x_3 &= -29 \\
4x_1 - x_2 + 11x_3 &= 23
\end{aligned}
$$

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 6 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ -16 \\ 11 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} 23 \\ 1 \\ -29 \\ 23 \end{bmatrix}$

Want to find $x_1, x_2, x_3 \in \mathbb{R}$ such that

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{u}.$$

Since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthogonal, can do this by finding dot products

$$\vec{u}\cdot\vec{v}_1 = 23\cdot 1 + 1\cdot 2 + (-29)\cdot 3 + 23\cdot 4 = 23+2-87+92 = 30$$

$$\vec{u}\cdot\vec{v}_2 = 23\cdot 6 + 1\cdot(-1) + (-29)\cdot 0 + 23\cdot(-1) = 138-1-23 = 114$$

$$\vec{u}\cdot v_3 = 23\cdot 2 + 1\cdot 1 + (-29)(-16) + 23\cdot 11 = 46+1+464+253 = 764$$

$$\vec{v}_1\cdot\vec{v}_1 = 1^2+2^2+3^2+4^2 = 1+4+9+16 = 30$$

$$\vec{v}_2\cdot\vec{v}_2 = 6^2+(-1)^2+0^2+(-1)^2 = 36+1+0+1 = 38$$

$$\vec{v}_3\cdot\vec{v}_3 = 2^2+1^2+(-16)^2+11^2 = 4+1+256+121 = 382$$

So
$$\vec{u} = \frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{u}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 + \frac{\vec{u}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3}\vec{v}_3$$

$$= \frac{30}{30}\vec{v}_1 + \frac{114}{38}\vec{v}_2 + \frac{764}{382}\vec{v}_3$$

$$= \vec{v}_1 + 3\vec{v}_2 + 2\vec{v}_3$$

Check:

$1+6\cdot 3+2\cdot 2 = 1+18+4 = 23$ ✓

$2\cdot 1-3+2 = 1$ ✓

$3\cdot 1-16\cdot 2 = 3-32 = -29$ ✓

$4\cdot 1-3+11\cdot 2 = 4-3+22 = 23$ ✓

And the solution to the original system of linear equations is:

$$\boxed{\begin{aligned} x_1 &= 1 \\ x_2 &= 3 \\ x_3 &= 2 \end{aligned}}$$