Eigenvectors of linear transformations on abstract vector spaces

The definitions of "eigenvectors" and "eigenvalues" still make sense for a linear transformation $T: V \to V$ on any vector space V (not just $\mathbb{R}^n$)

Definition If V is a vector space and $T: V \to V$ is a linear transformation then an eigenvector of T is a nonzero vector $\vec{v} \in V$ such that $T(\vec{v}) = \lambda \vec{v}$ for some scalar λ (called the eigenvalue)

① Let $C^\infty(\mathbb{R})$ denote the set of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Let $T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be the linear transformation defined by $T(f) = \frac{df}{dx} - 2f$. Which of the following are eigenvectors of T?

a) $\sin(x)$ Not an eigenvector

$T(\sin(x)) = \cos(x) - 2\sin(x)$

$\sin(0)=0$ so any scalar multiple of $\sin(x)$ must be 0 at 0. But $\cos(0) - 2\sin(0) = 1$ so $\cos(x) - 2\sin(x)$ is not a scalar multiple of $\sin(x)$

b) e^x

$T(e^x) = e^x - 2e^x = -e^x$

Eigenvector with eigenvalue -1

c) $5x + 2$ Not an eigenvector

$T(5x+2) = 5 - 2(5x+2) = -10x + 1$

This is not a scalar multiple of $5x+2$. To see why, evaluate both at $-2/5$

d) e^{2x} Eigenvector with eigenvalue 0

$T(e^{2x}) = 2e^{2x} - 2e^{2x} = 0$

Complex eigenvalues

Everything we've done in this class so far still works if we use complex numbers instead of real numbers.

Why care? Some matrices with <u>only real numbers</u> can only be diagonalized if we are allowed to use complex numbers.

<u>Example</u>: Rotation by 90° counterclockwise: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

No eigenvectors because it <u>rotates</u> every vector in $\mathbb{R}^2$, so it does not send any vector to a multiple of itself.

Characteristic polynomial: $(-t)^2 + 1 = t^2 + 1$ ← no real roots

But if we can use complex numbers then:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$$

Slogan: Complex eigenvalues = rotation

① Diagonalize $A = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}$

① Eigenvalues of A : $1+2i,\ 1-2i$

$$\chi_A(t) = \det \begin{bmatrix} 3-t & 4 \\ -2 & -1-t \end{bmatrix} = (3-t)(-1-t) - 4(-2)$$

$$= -3 + t - 3t + t^2 + 8$$

$$= t^2 - 2t + 5$$

roots : $\dfrac{2 \pm \sqrt{(-2)^2 - 4 \cdot 5}}{2} = \dfrac{2 \pm \sqrt{4 - 20}}{2}$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$= 1 \pm 2i$$

② Basis for E_{1+2i}: $\mathrm{Null}(A-(1+2i)I_2) = \mathrm{span}\left\{\begin{bmatrix}-1-i\\1\end{bmatrix}\right\}$

$$\begin{bmatrix}3-(1+2i) & 4\\ -2 & -1-(1+2i)\end{bmatrix} = \begin{bmatrix}2-2i & 4\\ -2 & -2-2i\end{bmatrix} \xrightarrow{R_1=\frac{1}{2-2i}R_1} \begin{bmatrix}1 & 1+i\\ -2 & -2-2i\end{bmatrix}$$

$$\left(\frac{1}{2-2i} = \frac{2+2i}{(2+2i)(2-2i)} = \frac{2+2i}{4+4} = \frac{1+i}{4}\right)$$

$$\xrightarrow{R_2=R_2+2R_1} \begin{bmatrix}1 & 1+i\\ 0 & 0\end{bmatrix} \qquad \begin{array}{l} x_1 = (-1-i)x_2 \\ x_2 \text{ free}\end{array}$$

Check: $\begin{bmatrix}3 & 4\\ -2 & -1\end{bmatrix}\begin{bmatrix}-1-i\\1\end{bmatrix} = \begin{bmatrix}-3-3i+4\\ 2+2i-1\end{bmatrix} = \begin{bmatrix}1-3i\\1+2i\end{bmatrix}$

$$= (1+2i)\begin{bmatrix}-1-i\\1\end{bmatrix}$$

Warning: Any scalar multiple of $\begin{bmatrix}-1-i\\1\end{bmatrix}$ is also an eigenvector, where "scalar" means "complex number." But it is not always easy to immediately tell that one vector is a complex number multiple of another

Basis for E_{1-2i}: $\mathrm{Null}(A-(1-2i)I_2) = \mathrm{span}\left\{\begin{bmatrix}-1+i\\1\end{bmatrix}\right\}$

Trick: Since A's entries are all <u>real</u> numbers, eigenvalues & eigenvectors come in conjugate pairs.

i.e. $1+2i$ is an eigenvalue, so $\overline{1+2i} = 1-2i$ is as well, $\begin{bmatrix}-1-i\\1\end{bmatrix}$ is an eigenvector with eigenvalue $1+2i$ so $\overline{\begin{bmatrix}-1-i\\1\end{bmatrix}} = \begin{bmatrix}-1+i\\1\end{bmatrix}$ is an eigenvector with eigenvalue $\overline{1+2i} = 1-2i$

③ $A = \begin{bmatrix}-1-i & -1+i\\1 & 1\end{bmatrix}\begin{bmatrix}1+2i & 0\\0 & 1-2i\end{bmatrix}\begin{bmatrix}-1-i & -1+i\\1 & 1\end{bmatrix}^{-1}$

If A is a 2×2 real matrix with complex eigenvalues then A is similar to a matrix of the form

$$r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

in other words, in some basis, A just looks like a scaling combined with a rotation.

If the complex eigenvalues are $a+bi$ and $a-bi$ then r and θ are such that

$$a+bi = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

↳ every complex number can be written in this form