Review

1. A 2×3 matrix

$\text{Null}(A) = \text{span}\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$

$A\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

a) $\text{rank}(A) = 2$

rank-nullity $\quad \text{rank}(A) + \dim(\text{Null}(A)) = 3$

$\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$ → free

pivot col.s

b) Find 3 sol'ns to $A\vec{x} = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad \begin{bmatrix}1\\2\\3\end{bmatrix} + t\begin{bmatrix}1\\2\\-1\end{bmatrix}$

$\begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}1\\2\\-1\end{bmatrix} = \begin{bmatrix}2\\4\\2\end{bmatrix}, \quad \begin{bmatrix}1\\2\\3\end{bmatrix} + 2\begin{bmatrix}1\\2\\-1\end{bmatrix} = \begin{bmatrix}3\\6\\1\end{bmatrix}$

$A\left(\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}1\\2\\-1\end{bmatrix}\right) = A\begin{bmatrix}1\\2\\3\end{bmatrix} + A\begin{bmatrix}1\\2\\-1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix}$

$= \begin{bmatrix}1\\0\end{bmatrix}$

Eigenvectors & Eigenvalues

Def If $T: V \to V$ is a linear transformation, an **eigenvector** of T is a nonzero vector $\vec{v} \in V$ such that $T(\vec{v}) = \lambda \vec{v}$ where λ is a scalar

eigenvalue for $\vec{v}$

Goal: We want to find a basis for V consisting only of eigenvectors for T

Why? The matrix for T in this basis is very simple e.g. $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Warning: This is not always possible

How to find eigenvectors & eigenvalues of an $n \times n$ matrix A?

(1) Find $\det(A - tI_n)$

characteristic polynomial of A $\chi_A(t)$

(2) Roots of $\chi_t(A)$ are the eigenvalues of A

(3) For each eigenvalue λ, Find $\mathrm{Null}(A - \lambda I_n)$ to find all eigenvectors w/ eigenvalue λ

① Which of the following are eigenvectors of $\begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}$

a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Yes b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ Yes c) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ No d) $\begin{bmatrix} -3 \\ -6 \end{bmatrix}$ Yes

$$\begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+1 \\ -2+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\cdot1+1\cdot2 \\ -2\cdot1+5\cdot2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eigenvectors must be nonzero

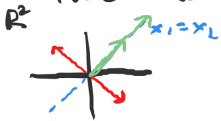
$$\begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ -6 \end{bmatrix} = \begin{bmatrix} 2\cdot-3+1\cdot-6 \\ -2\cdot-3+5\cdot-6 \end{bmatrix} = \begin{bmatrix} -12 \\ -24 \end{bmatrix} = 4 \cdot \begin{bmatrix} -3 \\ -6 \end{bmatrix}$$

② Suppose $\vec{v}_1, \vec{v}_2$ eigenvectors for T, both with eigenvalue $\underline{5}$. True/False: $3\vec{v}_1 - \vec{v}_2$ is also an eigenvector of T

False (because $3\vec{v}_1 - \vec{v}_2$ could be $\vec{0}$)

$$T(3\vec{v}_1 - \vec{v}_2) = 3T(\vec{v}_1) - T(\vec{v}_2) = 3\cdot(5\cdot\vec{v}_1) - 5\cdot\vec{v}_2$$
$$= 5(3\vec{v}_1 - \vec{v}_2)$$

③ Find an eigenvector of reflection across the line $x_1 = x_2$



$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigenvalue 1

$\begin{bmatrix} 2 \\ 2 \end{bmatrix} \to \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ eigenvalue -1

④ Find an eigenvector of $\begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 17 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 17 \\ 0 \\ 0 \end{bmatrix}$$

eigenvalue 17

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ldots$

⑤ Find eigenvalues & eigenvectors of $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

① $\det\left(\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} - t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 1-t & 2 \\ -1 & 4-t \end{bmatrix}$

$= (1-t)(4-t) - (-1)\cdot 2$

$= 4 - 5t + t^2 + 2$

$= t^2 - 5t + 6$

$= (t-2)(t-3)$

② Eigenvalues: 2, 3

③ <u>Eigenvalue 2</u>: $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$

$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 = -R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_2 free
$x_1 = 2x_2$

$\begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\qquad E_2 = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+2 \\ -2+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ✓

Eigenvalue 3

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 - \frac{1}{2}R_1} \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 = -\frac{1}{2}R_1}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

x_2 free

$x_1 = x_2$

$$\begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$E_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2 \\ -1+4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \checkmark$$

$$\boxed{\begin{array}{l} \text{Eigenvalues: } 2,\ 3 \\ E_2 = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \quad E_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \end{array}}$$