Review ① True or False: The set of invertible 3×3 matrices is a subspace of $M_{3\times3}$ (the space of all 3×3 matrices). False

Remember that there are three things to check to see if a subset W of a vector space V is a subspace of V: W must contain the zero vector of V, W must be closed under vector addition and W must be closed under scalar multiplication.

① Contains the zero vector of $M_{3\times3}$? No. $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ is not invertible.

② Closed under vector addition? No. $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ & $\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$ are invertible but $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ is not.

③ Closed under scalar multiplication? No. $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ is invertible but $0\cdot\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ is not.

Coordinates ①

$p(x) = -2x^2 + 4x + 4$

$q(x) = 3x^2 + 6x - 2$

$r(x) = -2x^2 + x + 3$

a) What is the dimension of $\text{span}\{p(x), q(x), r(x)\}$?

Method: Translate to $\mathbb{R}^3$ using a nice basis for $\mathbb{P}_2$ and solve there using row reduction.

Basis for $\mathbb{P}_2$: $B = \{x^2, x, 1\}$

$$[p(x)]_B = \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix} \quad [q(x)]_B = \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix} \quad [r(x)]_B = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

dimension of $\text{span}\{p(x), q(x), r(x)\}$ = rank of $\begin{bmatrix} -2 & 3 & -2 \\ 4 & 6 & 1 \\ 4 & -2 & 3 \end{bmatrix}$

$$\begin{bmatrix} -2 & 3 & -2 \\ 4 & 6 & 1 \\ 4 & -2 & 3 \end{bmatrix} \xrightarrow[R_3 = R_3 + 2R_1]{R_2 = R_2 + 2R_1} \begin{bmatrix} -2 & 3 & -2 \\ 0 & 12 & -3 \\ 0 & 4 & -1 \end{bmatrix} \xrightarrow{R_2 = \frac{1}{3}R_2} \begin{bmatrix} -2 & 3 & -2 \\ 0 & 4 & -1 \\ 0 & 4 & -1 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2}$$

$$\begin{bmatrix} -2 & 3 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{ REF}$$

2 pivots $\Rightarrow$ rank 2 $\Rightarrow$ dimension of $\text{span}\{p(x), q(x), r(x)\}$ is 2.

b) Find a basis for span $\{p(x), q(x), r(x)\}$

Recall that a basis for a vector space (or a subspace) is a list of vectors which span the whole space and which are linearly independent.

$p(x), q(x), r(x)$ span all of span$\{p(x), q(x), r(x)\}$ but we know from part (a) that they are not linearly independent. So we want to remove some to make them linearly independent. We can do this by translating to $\mathbb{R}^3$, row reducing, and just keeping the ones corresponding to pivot columns.

pivot columns

Row reduced matrix in $\mathbb{R}^3$ (from part (a)): $\begin{bmatrix} -2 & 3 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

corresponds to $q(x)$

corresponds to $p(x)$

So one basis for span$\{p(x), q(x), r(x)\}$ is $p(x), q(x)$

out of many possible bases

The matrix of a Linear Transformation.

① Let $T : \mathbb{P}_2 \longrightarrow \mathbb{R}^3$ be defined by

$$T(p) = \begin{bmatrix} \int_0^2 p(x)\,dx \\ \int_1^3 p(x)\,dx \end{bmatrix}$$

a) Let $B = \{1, x, x^2\}$. Let $C = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. What is ${}_C[T]_B$?

Remember that ${}_C[T]_B$ is the matrix representing T relative to the bases B and C. To find it, check what T does to each vector in B and write the results as coordinate vectors in C.

$$T(1) = \begin{bmatrix} \int_0^2 1\,dx \\ \int_1^3 1\,dx \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad T(x) = \begin{bmatrix} \int_0^2 x\,dx \\ \int_1^3 x\,dx \end{bmatrix} = \begin{bmatrix} x^2/2 \,\Big|_0^2 \\ x^2/2 \,\Big|_1^3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$T(x^2) = \begin{bmatrix} \int_0^2 x^2\,dx \\ \int_1^3 x^2\,dx \end{bmatrix} = \begin{bmatrix} x^3/3 \,\Big|_0^2 \\ x^3/3 \,\Big|_1^3 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 26/3 \end{bmatrix}$$

$$[T(1)]_C = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$[T(x)]_C = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$[T(x^2)]_C = \begin{bmatrix} 8/3 \\ 26/3 \end{bmatrix}$$

Since C is the standard basis for $\mathbb{R}^2$, for any vector $\vec{v} \in \mathbb{R}^2$, $[\vec{v}]_C = \vec{v}$ (e.g. to write $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we just write $\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so the coordinate vector is $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$).

This is a special feature of the standard basis and is not true of other bases for $\mathbb{R}^n$.

$$_C[T]_B = \begin{bmatrix} 2 & 2 & 8/3 \\ 2 & 4 & 26/3 \end{bmatrix}$$

b) Find a basis for range(T).

Recall that the range of T corresponds to the column space of any matrix representing T

We need to row reduce ${}_C[T]_B$.

$$\begin{bmatrix} 2 & 2 & 8/3 \\ 2 & 4 & 26/3 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 2 & 2 & 8/3 \\ 0 & 2 & 18/3 \end{bmatrix} \quad \text{REF}$$

A basis for $\mathrm{Col}({}_C[T]_B)$ is $\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Normally we would have to translate back from coordinate vectors in C to the actual vectors, but since C is the standard basis, we don't.

So one basis for range(T) is $\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$

(which actually means range(T) is all of $\mathbb{R}^2$ so any basis for $\mathbb{R}^2$ would work).

c) What is the dimension of the kernel of T?

Recall that the kernel of T corresponds to the null space of any matrix representing T.

Since $\operatorname{rank}({}_C[T]_B) = 2$, the null space has dimension 1 (by the rank-nullity theorem). So the kernel of T also has dimension 1.

d) Find a nontrivial (i.e. nonzero) element of the kernel of T.

kernel of $T \approx \operatorname{Null}({}_C[T]_B)$

$${}_C[T]_B = \begin{bmatrix} 2 & 2 & 8/3 \\ 2 & 4 & 26/3 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 2 & 2 & 8/3 \\ 0 & 2 & 18/3 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 2 & 0 & -10/3 \\ 0 & 2 & 18/3 \end{bmatrix}$$

$$\xrightarrow[R_2 = \frac{1}{2} R_2]{R_1 = \frac{1}{2} R_1} \begin{bmatrix} 1 & 0 & -5/3 \\ 0 & 1 & 3 \end{bmatrix}$$

Solutions to homogeneous equation:

$$x_1 = (5/3)x_3$$
$$x_2 = -3x_3$$
$$x_3 \text{ free}$$

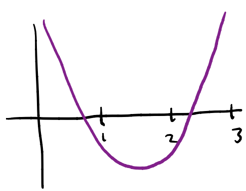
setting $x_3 = 3$ gives $\begin{bmatrix} 5 \\ -9 \\ 3 \end{bmatrix}$

Translating back to $\mathbb{P}_2$ gives

$$5 \cdot 1 - 9 \cdot x + 3 \cdot x^2 = 3x^2 - 9x + 5$$

one element of $\ker(T)$ (there are many others)

If you graph this polynomial, you will see it looks like this



The point is that its integral from 0 to 2 is 0, as is its integral from 1 to 3

(2) A quadratic polynomial is completely determined by its value on any three points. This can be shown using linear algebra.

a) Let $T : \mathbb{P}_2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}$$

Find $T(1)$, $T(x)$, $T(x^2)$

$$T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad T(x) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \qquad T(x^2) = \begin{bmatrix} 0^2 \\ 1^2 \\ 2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

b) Let $B = \{1, x, x^2\}$ $\quad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Find ${}_C[T]_B$

Recall that ${}_C[T]_B$ records what T does to each basis vector in B. To find it, we evaluate T on each vector in B and write the results as coordinate vectors in C. But since C is the standard basis for $\mathbb{R}^3$, the coordinate vector in C of any vector $\vec{v} \in \mathbb{R}^3$ is just $\vec{v}$ itself (see comments on problem 1)

So $${}_C[T]_B = \begin{bmatrix} 1 & 0 & 6 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

c) Check that T is invertible and find ${}_B[T^{-1}]_C$

Recall that T is invertible if and only if the matrix ${}_C[T]_B$ is invertible and if so, ${}_B[T^{-1}]_C = {}_C[T]_B^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - 2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right] \xrightarrow{R_3 = \frac{1}{2} R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right] \xrightarrow{R_2 = R_2 - R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3/2 & 2 & -1/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right]$$

So ${}_C[T]_B$ is invertible and

$${}_B[T^{-1}]_C = {}_C[T]_B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 2 & -1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}$$

d) Find a polynomial $p(x) \in \mathbb{P}_2$ such that

$p(0) = 10 \quad p(1) = 5 \quad$ and $\quad p(2) = -3$

This is equivalent to asking for $p(x) \in \mathbb{P}_2$ such that $T(p) = \begin{bmatrix} 10 \\ 5 \\ -3 \end{bmatrix}$. And finding a solution to $T(p) = \vec{v}$ is equivalent to solving ${}_C[T]_B \vec{x} = [\vec{v}]_C$

So we want to solve ${}_C[T]_B\,\vec{x} = \begin{bmatrix} 10 \\ 5 \\ -3 \end{bmatrix}$

We can do so by multiplying by ${}_C[T]_B^{-1}$

(i.e. we can find a solution to $T(p) = \begin{bmatrix} 10 \\ 5 \\ -3 \end{bmatrix}$ by applying T^{-1} to both sides).

$$\begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 2 & -1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ -7/2 \\ -3/2 \end{bmatrix}$$

Translating back to $\mathbb{P}_2$ gives $(-3/2)x^2 - (7/2)x + 10$

Moreover, a formula for the unique quadratic polynomial p s.t. $p(0) = a_0$, $p(1) = a_1$, $p(2) = a_2$ is:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 2 & -1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_0 \\ -3a_0/2 + 2a_1 - a_2/2 \\ a_0/2 - a_1 + a_2/2 \end{bmatrix}$$

$(a_0/2 - a_1 + a_2/2)x^2 + (-3a_0/2 + 2a_1 - a_2/2)x + a_0$