Review

1. True or False: The set of invertible 3×3 matrices is a subspace of $M_{3\times3}$ (i.e. of the vector space of all 3×3 matrices.

More Practice with Coordinates

1. Consider the following three polynomials in $\mathbb{P}_2$.

$$\begin{aligned} p(x) &= -2x^2 + 4x + 4 \\ q(x) &= 3x^2 + 6x - 2 \\ r(x) &= -2x^2 + x + 3 \end{aligned}$$

   (a) What is the dimension of $\text{span}\{p(x), q(x), r(x)\}$?
   (b) Find a basis for $\text{span}\{p(x), q(x), r(x)\}$.

The Matrix of a Linear Transformation Relative to a Basis

1. Let $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(p) = \begin{bmatrix} \int_0^2 p(x)\,dx \\ \int_1^3 p(x)\,dx \end{bmatrix}$$

   You do not need to check that T is a linear transformation.

   (a) Let $\mathcal{B}$ be the basis $1, x, x^2$ for $\mathbb{P}^2$ and let $\mathcal{C}$ be the basis $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ for $\mathbb{R}^2$ (i.e. the standard basis). Find ${}_{\mathcal{C}}[T]_{\mathcal{B}}$.
   (b) Find a basis for the range of T.
   (c) What is the dimension of the kernel of T?
   (d) Find a nontrivial element of the kernel of T. Try graphing the polynomial that you found.

2. You may have heard before that knowing the value of a quadratic polynomial on three points completely determines the polynomial. Let's use linear algebra to see how to do this.

   (a) Let $T\colon \mathbb{P}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}.$$

   Calculate $T(1), T(x)$, and $T(x^2)$.
   (b) Let $\mathcal{B}$ be the basis $1, x, x^2$ for $\mathbb{P}^2$ and let $\mathcal{C}$ be the standard basis for $\mathbb{R}^3$. Find ${}_{\mathcal{C}}[T]_{\mathcal{B}}$.
   (c) Check that T is invertible and find a matrix representing the inverse of T.
   (d) Use your answer to part (c) to find a polynomial p such that $p(0) = 10, p(1) = 5$, and $p(2) = -3$.
   (e) **Challenge problem:** Find a formula for the unique quadratic polynomial p such that $p(a_0) = b_0, p(a_1) = b_1$, and $p(a_2) = b_2$ (assuming that a_0, a_1, a_2 are all distinct).