Math 54 Final Review Solutions

Several of the questions below are from Professor Nadler's Fall 2014 final exam.

1. Find a solution to the following system of linear equations. (Hint: if you notice something about the columns of the matrix you can calculate the answer pretty quickly.)

$$\begin{array}{rcrcrcr} x_1 & + & 6x_2 & + & 2x_3 & = & 23 \\ 2x_1 & - & x_2 & + & x_3 & = & 1 \\ 3x_1 & & & - & 16x_3 & = & -29 \\ 4x_1 & - & x_2 & + & 11x_3 & = & 23 \end{array}$$

Row reduction will of course solve this problem. But note that the columns of the coefficient matrix are all orthogonal. Recall that if $v_1, \ldots, v_n$ are orthogonal vectors and u is in the subspace that they span then

$$u = \frac{\langle u, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \ldots + \frac{\langle u, v_n \rangle}{\langle v_n, v_n \rangle} v_n$$

Thus to find the solution to the above equation, we just need to compute dot products. Let $\mathbf{b}$ be the vector

$$\begin{bmatrix} 23 \\ 1 \\ -29 \\ 23 \end{bmatrix}$$

and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the columns of the coefficient matrix. Then

$$\begin{aligned} \mathbf{b} \cdot \mathbf{v}_1 &= 23 + 2 - 29 \cdot 3 + 23 \cdot 4 = 30 \\ \mathbf{b} \cdot \mathbf{v}_2 &= 23 \cdot 6 - 1 - 23 = 114 \\ \mathbf{b} \cdot \mathbf{v}_3 &= 23 \cdot 2 + 1 - 29 \cdot (-16) + 23 \cdot 11 = 764 \\ \mathbf{v}_1 \cdot \mathbf{v}_1 &= 1^2 + 2^2 + 3^2 + 4^2 = 30 \\ \mathbf{v}_2 \cdot \mathbf{v}_2 &= 6^2 + (-1)^2 + 0^2 + (-1)^2 = 38 \\ \mathbf{v}_3 \cdot \mathbf{v}_3 &= 2^2 + 1^2 + (-16)^2 + 11^2 = 4 + 1 + 256 + 121 = 382 \end{aligned}$$

Finally we need to check that the linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ that this gives us is actually equal to $\mathbf{b}$ (otherwise $\mathbf{b}$ is not in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and the system is not consistent). We have

$$\frac{30}{30}\mathbf{v}_1 + \frac{114}{38}\mathbf{v}_2 + \frac{764}{382}\mathbf{v}_3 = \begin{bmatrix} 23 \\ 1 \\ -29 \\ 23 \end{bmatrix}.$$

So the system is consistent and the solution is

$$x_1 = \frac{30}{30} = 1$$
$$x_2 = \frac{114}{38} = 3$$
$$x_3 = \frac{764}{382} = 2$$

2. Find a solution to $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 6 & 2 \\ 2 & -1 & 1 \\ 3 & 0 & -16 \\ 4 & -1 & 11 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 23 \\ 1 \\ -29 \\ 23 \end{bmatrix}$$

This is equivalent to the first question, so the solution is

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

(a) How many pivots does A have?

Since the columns are all nonzero and orthogonal, they are all linearly independent and thus each column has a pivot. So there are 3 pivots.

(b) What is $\dim(\operatorname{Null} A)$?

Since there are three pivots, there are no columns without pivots and thus the dimension of the null space is 0.

(c) What is $\operatorname{rank} A$?

Since there are three pivots, the rank is 3.

(d) What is $\dim(\operatorname{Row} A)$?

The dimension of the column space and the dimension of the row space of a matrix are always equal. Thus the dimension is 3.

(e) Is the linear transformation given by $\mathbf{v} \mapsto A\mathbf{v}$ one-to-one? Onto?

Since there is a pivot in every column of A it is one-to-one. Since there are 3 pivots, but 4 rows, not every row has a pivot. Thus it is no onto.

(f) Find a matrix B such that $BA = I$.

Recall that for any matrix C, the entry of C^TC in row i and column j is equal to the dot product of the i^{th} column and the j^{th} column of C. Since the columns of A are orthogonal, A^TA is diagonal and the entries on the diagonal are equal to the dot product of each column of A with itself. We already calculated these values in the first question, so we have

$$A^TA = \begin{bmatrix} 30 & 0 & 0 \\ 0 & 38 & 0 \\ 0 & 0 & 382 \end{bmatrix}$$

Inverting the diagonal matrix on the right gives us

$$\left(\begin{bmatrix} 1/30 & 0 & 0 \\ 0 & 1/38 & 0 \\ 0 & 0 & 1/382 \end{bmatrix} A^T \right) A = I$$

3. Find a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ that is equal to $\mathbf{b}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ -16 \\ 11 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 23 \\ 1 \\ -29 \\ 23 \end{bmatrix}$$

This is yet another rephrasing of the first question, so the solution is

$$\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 = \mathbf{b}$$

4. For which pair of real numbers a and b is the following matrix rank one:

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & a & 1 \\ 3 & -6 & b \end{bmatrix}$$

A matrix is rank one when its column space has dimension one or, equivalently, when any nonzero column spans the entire column space. So for the above matrix to be rank one, every column must be in the span of the first column, and hence a scalar multiple of the first column. By looking at the first entry of each column, we see that the second column must be -2 times the first column and the third column must be -1 times the first column. So a must be 2 and b must be -3.

5. Let A be a 2×2 matrix with real entries and eigenvalues 2 and -3. What is the characteristic polynomial of A? What if 2 is the only eigenvalue? What if $2 - 2i$ is an eigenvalue?

If α is an eigenvalue of a matrix then $(\lambda - \alpha)$ must be a factor of the characteristic polynomial. Also, the leading term of the characteristic polynomial of an $n \times n$ matrix is always $(-1)^n \lambda^n$. So the answer to the first question is $(2-\lambda)(-3-\lambda) = \lambda^2 + \lambda - 6$. Also, if 2 is the only eigenvalue then $(2 - \lambda)$ is the only possible factor and since the characteristic polynomial of a 2×2 matrix is always degree 2 and since every polynomial factors into linear terms over the complex numbers, the answer to the second question must be $(2 - \lambda)^2 = \lambda^2 - 4\lambda + 4$. Finally, if α is a complex number that is an eigenvalue of a real matrix then $\overline{\alpha}$ is also an eigenvalue. Thus the answer to the third question is $(2 - 2i - \lambda)(2 + 2i - \lambda) = \lambda^2 - 4\lambda + 8$.

6. Find a nontrivial solution $u(x, t)$ to the heat equation $(\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial^2 x})$ with the usual boundary values $(u(0, t) = u(L, t) = 0)$ such that $u(L/27, 0) = 0$.

In general we can solve this problem by finding any nonzero function f such that $f(L/27) = 0$ and solving the heat equation with f as the initial condition. Recall that this would involve writing f as a sum of functions of the form $\sin\left(\frac{n\pi x}{L}\right)$. To make our lives easier, let's see if we can choose f to be equal to $\sin\left(\frac{n\pi x}{L}\right)$ for some n. And in fact we can! There are many possibilities, but one is $f(x) = \sin\left(\frac{27\pi x}{L}\right)$. Solving the heat equation with this as the initial condition gives us

$$u(x, t) = e^{-\beta(\frac{27\pi}{L})^2 t} \sin\left(\frac{27\pi x}{L}\right)$$

Note that this is just one of many possible answers to this question.

7. Let $f\colon [0, L] \to \mathbb{R}$ be a function such that $f(0) = f(L) = 0$ and f' is continuous. Suppose that the n^{th} coefficient of the fourier sine series for f is $\frac{1}{2^n}$.

   (a) Find $f(\frac{L}{2})$

   The function f defined above satisfies the condition given in class that guarantees that the fourier series converges everywhere to f. Thus we have

$$f\left(\frac{L}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sin\left(\frac{n\pi L}{2L}\right)$$
$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \sin\left(\frac{n\pi}{2}\right)$$

Now note that $\sin\left(\frac{n\pi}{2}\right)$ is 0 if n is even. And if $n = 2k+1$ then $\sin\left(\frac{n\pi}{2}\right)$ is 1 if k is even and -1 if k is odd. Thus the sum above is equal to

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{2^{2k+1}} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k$$

This is just a geometric series, so it is equal to

$$\frac{1}{2}\left(\frac{1}{1 - \frac{-1}{4}}\right) = \frac{2}{5}$$

(b) Find the fourier sine series for $f(x)\cos\left(\frac{\pi x}{L}\right)$. Hint: use the fact that for any A and B, $\sin(A)\cos(B) = \frac{1}{2}(\sin(A-B) + \sin(A+B))$.

The n^{th} term of the fourier sine series of $f(x)\cos\left(\frac{\pi x}{L}\right)$ is

$$\frac{\langle f(x)\cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\rangle}{\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\rangle}$$

Using the fact provided in the hint,

$$f(x)\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \frac{f(x)}{2}\left(\sin\left(\frac{(n-1)\pi x}{L}\right) + \sin\left(\frac{(n+1)\pi x}{L}\right)\right)$$

Thus we have

$$\begin{aligned}\left\langle f(x)\cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\rangle &= \int_0^L f(x)\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{1}{2}\int_0^L f(x)\Bigg(\sin\left(\frac{(n-1)\pi x}{L}\right) \\ &\qquad + \sin\left(\frac{(n+1)\pi x}{L}\right)\Bigg)\,dx \\ &= \frac{1}{2}\Bigg(\int_0^L f(x)\sin\left(\frac{(n-1)\pi x}{L}\right)\,dx \\ &\qquad + \int_0^L f(x)\sin\left(\frac{(n+1)\pi x}{L}\right)\,dx\Bigg) \\ &= \frac{1}{2}\Bigg(\left\langle f(x), \sin\left(\frac{(n-1)\pi x}{L}\right)\right\rangle \\ &\qquad + \left\langle f(x), \sin\left(\frac{(n+1)\pi x}{L}\right)\right\rangle\Bigg)\end{aligned}$$

When n is greater than 1 this means that the n^{th} term of the fourier sine series for $f(x)\cos\left(\frac{\pi x}{L}\right)$ is simply $\frac{1}{2}$ times the sum of the $(n+1)^{\text{th}}$ and $(n-1)^{\text{th}}$ terms of the sine fourier series for $f(x)$–i.e. $\frac{1}{2}\left(\frac{1}{2^{n-1}} + \frac{1}{2^{n+1}}\right) = \frac{5}{2^{n+2}}$. When $n = 1$, since $\sin(0) = 0$ we just get $\frac{1}{2}$ times the 2^{nd} term of the fourier sine series for $f(x)$–i.e. $\frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}$.

8. The following assertions are FALSE. For each one, provide a counterexample to show it is false.

   (a) If A is a 2×2 symmetric matrix with positive integer entries then any eigenvalue of A is positive or zero.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   (b) Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a one-to-one linear transformation and let $\mathcal{B}$ be a basis for $\mathbb{R}^3$. Then there is a basis $\mathcal{C}$ of $\mathbb{R}^2$ such that

$${}_{\mathcal{B}}[T]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Let $\mathcal{B}$ be the standard basis for $\mathbb{R}^3$ and let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Then T is one-to-one because its standard matrix has a pivot in every column. On the other hand, for any basis $\mathcal{C}$ of $\mathbb{R}^2$ we have ${}_{\mathcal{B}}[T]_{\mathcal{C}} = AP_{\mathcal{C}}$. Since the first two rows of A are just the 2×2 identity matrix, this means that the first two rows of ${}_{\mathcal{B}}[T]_{\mathcal{C}}$ must be equal to $P_{\mathcal{C}}$. So in order for the first two rows of ${}_{\mathcal{B}}[T]_{\mathcal{C}}$ to be the 2×2 identity matrix we must have $P_{\mathcal{C}} = I$ and so ${}_{\mathcal{B}}[T]_{\mathcal{C}} = A$, which is not equal to the matrix in the question.

(c) Let f and g be two differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. If the Wronskian $W[f,g](t)$ is always 0 then f and g are linearly dependent.

This solution comes from a homework problem. Let $f(t) = t^2$ and $g(t) = t|t|$. Then

$$W[f,g](t) = \det \begin{bmatrix} t^2 & t|t| \\ 2t & 2|t| \end{bmatrix} = 0$$

but f and g are linearly independent. This example shows that many nice properties of the Wronskian that hold when the functions are all solutions to the same linear ODE *do not hold in general*.

(d) If the columns of a matrix are orthonormal then so are the rows (hint: this is true if the matrix is square).

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}$$

9. Mark each of the following statements true or false. For each statement, either give a proof that it is always true or give a counterexample to show it can be false.

(a) For any inner product space V, if vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy $||\mathbf{u}|| = 1, ||\mathbf{v}|| = 1$ and $||\mathbf{u} - \mathbf{v}|| = \sqrt{2}$ then $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

True. We have

$$\begin{aligned} 2 &= ||\mathbf{u} - \mathbf{v}||^2 \\ &= \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle - 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= 1 - 2\langle \mathbf{u}, \mathbf{v} \rangle + 1 \\ &= 2 - 2\langle \mathbf{u}, \mathbf{v} \rangle. \end{aligned}$$

Thus $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ so $\mathbf{v}$ and $\mathbf{u}$ are orthogonal.

(b) If A is symmetric and U is orthogonal then UAU^{-1} is symmetric.

True. Since U is orthogonal, $U^{-1} = U^T$. Thus we have

$$\begin{aligned}(UAU^{-1})^T &= (UAU^T)^T \\ &= (U^T)^T A^T U^T \\ &= UAU^T \\ &= UAU^{-1}\end{aligned}$$

(c) If the characteristic polynomial of an $n \times n$ matrix A has a nonzero constant coefficient then A is invertible.

True. If a square matrix A is not invertible, then 0 must be an eigenvalue, and hence a root of the characteristic polynomial. Thus the characteristic polynomial can be written as $(\lambda - 0)p(\lambda) = \lambda p(\lambda)$ for some polynomial p. And the constant coefficient of $\lambda p(\lambda)$ is 0.

(d) Let V be the vector space of differentiable functions on $\mathbb{R}$. The linear transformation $\frac{d^2}{dt^2} - 2\frac{d}{dt} + 2I$ is one-to-one.

False. Any solution to the differential equation $y'' - 2y' + y = 0$ will be in the kernel of T. Since this differential equation has nonzero solutions, the kernel of T is no just $\{0\}$ and so T is not one-to-one.