Midterm 2

Math 114S, Winter 2022

Instructions. Turn in your exam on Gradescope by 10 am PST on Saturday, February 26th. Late exams are not accepted, so I advise you to turn it in at least a few minutes early. You may consult the lecture notes, course textbooks or standard websites such as Wikipedia. **You may not attempt to search for specific exam questions online nor may you communicate with anyone besides the instructors of the course about the contents of the exam.** In particular, you should not talk to your fellow students about the exam, except to ask logistics questions, and you may not post exam-related questions on any Q&A websites or forums. If you have any questions about the exam, please make a private post on piazza. There are 50 points in total.

Note: For each problem below, unless otherwise stated, you may use the Axiom of Choice (including any of its consequences which we proved in class).

Short Answer Questions.

Question 1 (4 points)

Compute the rank of the following set: $A = \{\{\langle \mathbb{R}, \mathbb{Z}\rangle, \{\langle \mathbb{Q}, \omega\rangle\}, \langle \mathbb{R}, V_{\omega+2}\rangle\}\}$. You may take as given the calculations of ranks from the Homework 6 solutions. You should show work justifying your answer but you do not need to provide a formal proof that your answer is correct.

Question 2 (8 points)

For any subset $A \subseteq \omega$ and natural number n, say that n is *prime-in-A* if every divisor of n which is contained in A is equal to either 1 or n. For example, if A is the set of numbers greater than 10 then 14 is prime-in-A.

Define a class function $F\colon \mathbf{Ord} \to \mathcal{P}(\omega)$ by transfinite recursion as follows.

Zero case: $F(0) = \omega \setminus \{0, 1\}$

Successor case: $$F(\alpha+1) = \begin{cases} F(\alpha) \setminus \{n\} & \text{if } n \text{ is the least element of } F(\alpha) \text{ which is prime-in-}F(\alpha) \\ F(\alpha) & \text{if no such } n \text{ exists} \end{cases}$$

Limit case: $$F(\beta) = \bigcap_{\alpha<\beta} F(\alpha).$$

For both parts below, you should give a brief justification of your answer, but you do not need to provide a formal proof.

(a) What is $F(\omega + \omega)$?

(b) What is $F(\omega^3)$? Recall that ω^3 denotes the unique ordinal with order type $\omega \times \omega \times \omega$.

Question 3 (12 points)

For each set below, write either "countable," "continuum" or "other" to indicate, respectively, that the set is countable, has cardinality $|2^\omega|$ or that neither of those two possibilities hold. You do not need to provide any justification for your answers.

(a) The set of surjective functions $\omega \to \{0, 1, 2, \ldots, 10\}$.

(b) The set of surjective functions $\omega \to \omega$.

(c) The set of surjective functions $\omega \to 2^\omega$.

(d) The set of surjective functions $2^\omega \to 2^\omega$.

(e) The set of isomorphism classes of finite partial orders which are in $V_{\omega+5}$.

(f) $((2^\omega \times 2^\omega) \sqcup 2^\omega)^\omega$.

Long Answer Questions.

Question 4 (8 points)

If $f, g\colon \omega \to \omega$, then f *dominates* g if for all $n \in \omega$, $g(n) \leq f(n)$. If A is a set of functions from ω to ω then A is *dominating* if for every function $g\colon \omega \to \omega$ there is some $f \in A$ which dominates g. Prove that there is no countable dominating set of functions from ω to ω.

Hint: Diagonalization.

Question 5 (10 points)

Recall that a *graph* is an ordered pair $\langle V, E\rangle$ consisting of a set V, called the *set of vertices*, and a set E of ordered pairs of elements of V, called the *set of edges*. A subset $U \subset V$ is a *clique*[1] if for every $a, b \in U$, $\langle a, b\rangle \in E$. A clique U is *maximal* if there is no clique U' such that U is a proper subset of U'. Show that every graph has a maximal clique.

Question 6 (8 points)

Graphs $\langle V, E\rangle$ and $\langle V', E'\rangle$ are *isomorphic* if there is a bijection $f\colon V \to V'$ such that for all $u, v \in V$,

$$\langle u, v\rangle \in E \iff \langle f(u), f(v)\rangle \in E'.$$

Without using the Axiom of Foundation, show that for every graph $\langle V, E\rangle$, there is some $\alpha \in \mathbf{Ord}$ such that V_α contains a graph isomorphic to $\langle V, E\rangle$.

[1] This definition of graph and clique are slightly different from the standard definitions—normally cliques are only defined for undirected graphs without self-loops. Note that in the definition above, if U is a clique and $a, b \in U$ then we must have all of $\langle a, a\rangle$, $\langle a, b\rangle$ and $\langle b, a\rangle$ in U. I used this definition to keep the definitions as short as possible.