Midterm 1 Solutions

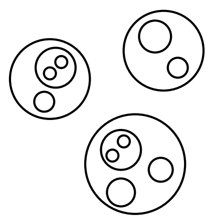
Math 114S, Winter 2022

Instructions. Turn in your exam on Gradescope by 10 am PST on Saturday, January 29th. Late exams are not accepted, so I advise you to turn it in at least a few minutes early. You may consult the lecture notes, course textbooks or standard websites such as Wikipedia. You may not attempt to search for specific exam questions online nor may you communicate with anyone besides the instructors of the course about the contents of the exam (this includes your fellow classmates and also all online Q&A websites, etc). If you have any questions about the exam, please make a private post on piazza. There are 50 points in total.

Short Answer Questions.

Question 1 (5 points)

Consider the drawing below.



Let X be the set of circles in the drawing and define a binary relation $\sim$ on X by

$$x \sim y \iff x \text{ and } y \text{ contain the same number of circles.}$$

The relation $\sim$ is actually an equivalence relation on X (you do not need to prove this). How many equivalence classes does it have and how many elements does each equivalence class have? You do not need to justify your answer.

Solution: There are $\boxed{4}$ equivalence classes, consisting of those circles which contain $0, 2, 4$ and 5 circles, respectively. The sizes of the equivalence classes are $\boxed{9,\ 3,\ 1 \text{ and } 1.}$

Common Mistakes: Some people wrote things along the lines of "$[0]_\sim$ contains 9 elements." However, this seems to imply that 0 is an *element* of the equivalence class of circles which contain 0 circles, which is not correct. That equivalence class is a set of cricles and does not contain any numbers (however, 0 might be a reasonable *name* or *shorthand* for that equivalence class, but then the notation $[\cdot]_\sim$ does not really make sense).

Question 2 (5 points)

Using the definitions of ω and $\mathbb{Q}$ given in class, is there some natural number n such that

$$\underbrace{\bigcup \cdots \bigcup}_{n \text{ times}} \mathbb{Q} = \omega?$$

If so, what is it? You should briefly justify your answer, but you don't need to formally prove it is correct.

Hint: First try to figure out if there is some number of times you can take the union of $\mathbb{Q}$ to get $\mathbb{Z}$.

Solution: Yes, there is such an n and it is equal to $\boxed{6.}$ In other words:

$$\bigcup\bigcup\bigcup\bigcup\bigcup\bigcup\mathbb{Q} = \omega.$$

To see why this is valid, note that

- Elements of $\mathbb{Q}$ are subsets of $\mathbb{Z} \times (\mathbb{Z} \setminus \{0_\mathbb{Z}\})$ and every element of $\mathbb{Z} \times (\mathbb{Z} \setminus \{0_\mathbb{Z}\})$ is contained in some element of $\mathbb{Q}$ and thus $\bigcup\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0_\mathbb{Z}\})$.
- Elements of $\mathbb{Z} \times (\mathbb{Z} \setminus \{0_\mathbb{Z}\})$ are (formally) sets of sets of elements of $\mathbb{Z}$ so $\bigcup(\mathbb{Z} \times (\mathbb{Z} \setminus \{0_\mathbb{Z}\}))$ consists of sets of elements of $\mathbb{Z}$. Furthermore, every element of $\mathbb{Z}$ is contained in at least one of these sets because for any $n \in \mathbb{Z}$, $\{\{n\}, \{n, 1_\mathbb{Z}\}\} \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0_\mathbb{Z}\})$. So $\bigcup\bigcup(\mathbb{Z} \times (\mathbb{Z} \setminus \{0_\mathbb{Z}\})) = \mathbb{Z}$. Therefore $\bigcup\bigcup\bigcup\mathbb{Q} = \mathbb{Z}$.
- By similar reasoning, $\bigcup\bigcup\bigcup\mathbb{Z} = \omega$ and therefore $\bigcup\bigcup\bigcup\bigcup\bigcup\bigcup\mathbb{Q} = \omega$.

Comment: As one student pointed out, $\bigcup\omega = \omega$ and so actually any $n \geq 6$ is a valid answer to this question.

Question 3 (5 points)

Rewrite the formula $\langle 1, 1\rangle \in x$ in the language of set theory. You do not need to justify your answer.

In other words, find a formula with one free variable, x, which only contains variables, quantifiers ($\forall$ and $\exists$), logical symbols ($\neg, \wedge, \vee, \implies, \iff$), parentheses and the symbols $\in$ and $=$ and which is equivalent to the formula $\langle 1, 1\rangle \in x$.

Solution: There are multiple valid answers to this question. Here is one:

$$\begin{aligned}\exists x_1 \exists x_2 \exists x_3 \exists x_4 \big(\forall y_1 (\neg(y_1 \in x_1)) \wedge \forall y_2 (y_2 \in x_2 &\iff y_2 = x_1)\\ \wedge \forall y_3 (y_3 \in x_3 &\iff y_3 = x_2)\\ \wedge \forall y_4 (y_4 \in x_4 &\iff y_4 = x_3) \wedge x_4 \in x\big).\end{aligned}$$

Informally, the part of the formula after the existential quantifiers asserts that $x_1 = \varnothing$, $x_2 = \{\varnothing\}$, $x_3 = \{\{\varnothing\}\}$ and $x_4 = \{\{\{\varnothing\}\}\}$. So informally, the formula asserts that $\{\{\{\varnothing\}\}\} \in x$. To see why this is equivalent to $\langle 1, 1\rangle \in x$, note that by the definitions we gave in class

$$\langle 1, 1\rangle = \{\{1\}, \{1, 1\}\} = \{\{1\}, \{1\}\} = \{\{1\}\} = \{\{0^+\}\} = \{\{\varnothing^+\}\} = \{\{\varnothing \cup \{\varnothing\}\}\} = \{\{\{\varnothing\}\}\}.$$

Common Mistakes: Some students wrote something along the lines of

$$\forall y\, ((\ldots) \iff y \in x)$$

(where the three dots should be replaced with some formula asserting that $y = \langle 1, 1\rangle$). However, this formula is not quite right because it actually implies that $\langle 1, 1\rangle$ is the *only* element of x rather than just *an* element of x.

Another common mistake was to use $\implies$ instead of $\iff$ in the formula, i.e. in the formula in the solution to replace clauses like

$$\forall y_3\, (y_3 \in x_3 \iff y_3 = x_2)$$

with

$$\forall y_3\,(y_3 \in x_3 \implies y_3 = x_2)$$

Unfortunately this does not work because it is satisfied if x_3 is either $\{x_2\}$ *or* $\varnothing$.

Some people also thought that $\langle 1, 1\rangle$ is equal to $\{\{\varnothing\}\}$ instead of $\{\{\{\varnothing\}\}\}$.

Several people also used symbols which are not part of the language of set theory, such as 1 or $\varnothing$ or $\{$.

Question 4 (12 points)

Mark each of the following as True or False. You do not need to provide any justification for your answers.

(a) There is a set containing all binary relations with domain $\varnothing$.

Solution: $\boxed{\text{True.}}$ The only binary relation with domain $\varnothing$ is the relation $\varnothing$ and it is easy to prove in ZFC that $\{\varnothing\}$ is a set.

(b) Russell's paradox does not depend on the Axiom of Extensionality.

Solution: $\boxed{\text{True.}}$ The Axiom of Extensionality is not used anywhere in the construction of the Russell set (only naive comprehension is needed) or in the proof that the existence of such a set leads to a contradiction.

(c) Suppose X is a set, $f\colon X \to X$ is a function and A is a subset of X. Define $H\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by $H(B) = B \setminus f[B]$. If $h\colon \omega \to \mathcal{P}(X)$ is the function given by the Recursion Theorem, satisfying

$$h(0) = A \text{ and for all } n \in \omega,\ h(n+1) = H(h(n)),$$

then $h(100) = h(101)$.

Solution: $\boxed{\text{True.}}$ Let's calculate the first few values of h. By definition, $h(0) = A$ and $h(1) = A \setminus f[A]$. Now observe that since $A \setminus f[A] \subseteq A$, $f[A \setminus f(A)] \subseteq f[A]$. Since $A \setminus f[A]$ and $f[A]$ are disjoint, this implies that $A \setminus f[A]$ and $f[A \setminus f[A]]$ are also disjoint and thus $(A \setminus f[A]) \setminus (f[A \setminus f[A]]) = A \setminus f[A]$. Thus $h(2) = h(1) = A \setminus f[A]$. By the same reasoning, $h(3) = A \setminus f[A]$ as well and, more generally, one can show by induction that for all $n \geq 1$, $h(n) = A \setminus f[A]$.

Comment: This seems to have been the most frequently missed of the True/False questions.

(d) According to our definition of $\mathbb{R}$ in class, the set $\{q \in \mathbb{Q} \mid \exists p \in \mathbb{Q}\,(p \times q < 2)\}$ is a real number.

Solution: $\boxed{\text{False.}}$ The set is actually equal to all of $\mathbb{Q}$ (if $q < 2$ then $q \times 1 < 2$ and if $q \geq 2$ then $q \times (-1) < 2$) and so it is not a Dedekind cut.

Long Answer Questions.

Question 5 (8 points)

Let X be a set and A and B be subsets of X. Define a function $G\colon \{0,1\}^X \to \{0,1\}^A \times \{0,1\}^B$ by

$$G(f) = \langle f \restriction A, f \restriction B\rangle.$$

Prove that G is surjective if and only if $A \cap B = \varnothing$.

Recall that $\{0,1\}^X$ denotes the set of functions from X to $\{0,1\}$ and $f \restriction A$ denotes the restriction of f to A.

Solution: ($\Longrightarrow$) For this direction, it is easier to prove the contrapositive. Namely, assume that $A \cap B \neq \varnothing$ and we will show that G is not surjective. Define $f\colon A \to \{0,1\}$ to be the constantly 1 function and $g\colon B \to \{0,1\}$ to be the constantly 0 function. In other words:

$$\text{for all } x \in A,\ f(x) = 1$$
$$\text{and for all } x \in B,\ g(x) = 0.$$

We claim that $\langle f, g\rangle$ is not in the range of G and thus G is not surjective. To see why, let h be any function $X \to \{0,1\}$. We need to show that $G(h) \neq \langle f, g\rangle$. Let x be some element of $A \cap B$ (which must exist since we assumed the intersection is nonempty). Note that either $h(x) = 0$ or $h(x) = 1$. In the first case, $h \restriction A \neq f$ because they disagree on x and in the second case, $h \restriction B \neq g$ because they disagree on x. Therefore, in either case $G(h) = \langle h \restriction A, h \restriction B\rangle \neq \langle f, g\rangle$.

($\Longleftarrow$) Suppose that $A \cap B = \varnothing$ and let $f\colon A \to \{0,1\}$ and $g\colon B \to \{0,1\}$ be functions. We need to show that $\langle f, g\rangle$ is in the range of G. Define a binary relation $h \subseteq X \times \{0,1\}$ by

$$h = f \cup g \cup \{\langle x, 0\rangle \mid x \in X \setminus (A \cup B)\}.$$

It is clear that the domain of h is X. Also, h is a function because it is the union of three functions with disjoint domains. We now claim $G(h) = \langle f, g\rangle$. To show this we need to show that for any $x \in A$, $h(x) = f(x)$ and for any $x \in B$, $h(x) = g(x)$. This is true by construction of h so we are done.

Comment: Originally there was a typo in this question and $G(f)$ was defined as $(f \restriction A, f \restriction B)$. However, this does not match the notation for ordered pairs we established in class, which uses angle brackets rather than parentheses.

Question 6 (15 points)

In class we claimed that all known math can be formalized within set theory. Suppose that one day you meet some aliens from the planet Ksnadge who are skeptical of this claim. They tell you about a mathematical object that is very important to their development of mathematics—something they call the "Owezrd system." The Owezrd system consists of a set X, an element $a \in X$ and functions $L\colon X \to X$ and $R\colon X \to X$ which satisfy the following axioms:

(1) $a \notin \operatorname{range}(L)$ and $a \notin \operatorname{range}(R)$

(2) R and L are injective

(3) $\operatorname{range}(L)$ and $\operatorname{range}(R)$ are disjoint

(4) For all $A \subseteq X$, if $a \in A$ and A is closed under R and L (i.e. for all $x \in A$, $R(x) \in A$ and $L(x) \in A$) then $A = X$.

You want to show the Ksnadgeans that it is possible to construct an object that satisfies the axioms of the Owezrd system within set theory. In this problem you will see how to do this.

Comment: You might recognize the "Owezrd system" of the Ksnadgeans as an infinite complete binary tree. And in fact, "owezrd" backwards is "drzewo," which is the Polish word for "tree" ("Ksnadge" backwards is (almost) "Gdansk," a city in Poland).

(a) Let X be the set of functions of the form $f\colon n \to \{0,1\}$ for some $n \in \omega$. Show formally that we can form this set in ZFC set theory (recall that with our definition of ω, $0 = \varnothing$ and $n+1 = \{0,1,\ldots,n\}$ for any $n \in \omega$).

Solution: The key point is that if n is any element of ω then since $n \subseteq \omega$ any function $f\colon n \to \{0,1\}$ is a subset of $\omega \times \{0,1\}$. Thus we can construct X using the Axiom of Separation as follows:

$$X = \{f \in \mathcal{P}(\omega \times \{0,1\}) \mid f \text{ is a function and for some } n \in \omega,\ \mathrm{dom}(f) = n\}.$$

Common Mistakes: Many people tried to justify the existence of X by first showing that for each $n \in \omega$, the set $X_n := \{f \mid f\colon n \to \{0,1\}\}$ exists and then stating that $X = \bigcup_{n\in\omega} X_n$. The problem with this is that we can only construct the union $\bigcup_{n\in\omega} X_n$ if we know that the function

$$n \mapsto X_n$$

exists. Later in the course we will see how to prove this easily using an axiom of ZFC called the Axiom of Replacement. But for now we have no good way to prove that this function exists.

(b) Let $a = \varnothing$ be the empty function and let $L\colon X \to X$ and $R\colon X \to X$ be the functions

$$L(f) = f \cup \{\langle \mathrm{dom}(f), 0\rangle\}$$
$$R(f) = f \cup \{\langle \mathrm{dom}(f), 1\rangle\}.$$

Prove that range(L) and range(R) are disjoint. You may use without proof any standard facts about the natural numbers (i.e. ω).

Comment. If you are confused about the definition of L and R, note that if $f\colon n \to \{0,1\}$ then $L(f)$ and $R(f)$ are both functions from $n+1 = \{0,1,\ldots,n\}$ to $\{0,1\}$, they both agree with f on all inputs less than n, $L(f)(n) = 0$, and $R(f)(n) = 1$.

Solution: Suppose for contradiction that f is a function which is in the range of both L and R—i.e. there are functions g_1 and g_2 in X such that $L(g_1) = f$ and $R(g_2) = f$. There are two cases.

Case 1: $\mathrm{dom}(g_1) = \mathrm{dom}(g_2)$. Let $n = \mathrm{dom}(g_1) = \mathrm{dom}(g_2)$. Then $\langle n, 0\rangle \in L(g_1) = f$ and $\langle n, 1\rangle \in R(g_2) = f$. But this contradicts the assumption that f is a function.

Case 2: $\mathrm{dom}(g_1) \neq \mathrm{dom}(g_2)$. Let $n = \mathrm{dom}(g_1)$ and $m = \mathrm{dom}(g_2)$. Since $\langle n, 0\rangle L(g_1) = R(g_2) = g_2 \cup \{\langle m, 1\rangle\}$ and $n \neq m$, we must have $n \in \mathrm{dom}(g_2) = m$. And likewise, we must have $m \in \operatorname{dom}(g_1) = n$. But we mentioned in class that for $a, b \in \omega$, $a < b$ if and onl y if $a \in b$ so this means that $n < m$ and $m < n$, which contradicts the fact that $\leq$ is a total order on ω.

(c) Prove that $\langle X, a, L, R \rangle$ as defined in parts (a) and (b) of this question satisfies axiom (4) of the Owezrd system.

Solution: Suppose A is a subset of X such that $a \in A$ and A is closed under L and R. We will show by induction that for every $n \in \omega$, all functions in X with domain n are in A.

Base Case: Since $0 = \varnothing$, the only function with domain 0 is the empty function, $\varnothing$. And $\varnothing \in A$ by assumption.

Inductive Case: Assume for induction that all functions in X with domain n are in A and we will show that this also holds for functions in X with domain $n + 1$. Let $f\colon (n+1) \to \{0, 1\}$ by a function in X. Note that $\operatorname{dom}(f \restriction n) = n$ and so $f \restriction n \in A$ by the inductive assumption. Also note that $f = (f \restriction n) \cup \{\langle n, f(n) \rangle\}$. There are two cases to consider.

Case 1: $f(n) = 0$. In this case, $f = (f \restriction n) \cup \{\langle n, 0 \rangle\} = L(f \restriction n)$ and hence $f \in A$.

Case 2: $f(n) = 1$. In this case, $f = (f \restriction n) \cup \{\langle n, 1 \rangle\} = R(f \restriction n)$ and hence $f \in A$.

Common Mistakes: Some people tried to prove that $A = X$ by stating that any function $f \in X$ can be built from $a = \varnothing$ by a finite sequence of applications of L and R. This is the correct intuition but in this class we are trying to see how to formally build up math within set theory so this sort of proof is a bit too informal and the way to formalize this intuitive proof is to use induction.