Math 10B, Quiz 9 Solutions

1. (9 points) Suppose you consume some alcohol. Your body (specifically your liver) will naturally metabolize the alcohol and turn it into other substances. The rate at which your body metabolizes alcohol is proportional to the amount of alcohol currently in your system. In other words, the rate at which your body gets rid of alcohol is proportional to the amount of alcohol in your body.

   (a) Suppose you are drinking alcohol at a rate of 2 milliliters per second. Let $A(t)$ represent the number of milliliters of alcohol in your body at time t (where t is measured in seconds). Write a differential equation satisfied by A.

   Solution: Almost any time you need to write a differential equation to express how the quantity of some substance changes over time, the differential equation will have the form

   $$\text{derivative of the amount of the substance over time} = (\text{rate in}) - (\text{rate out}).$$

   In this problem, the substance is the alcohol in your body. The rate in is simply 2 mL/second and the rate out is proportional to the amount in your body, i.e. $kA(t)$ where k is some constant. Putting this all together, we get

   $$\boxed{\frac{dA}{dt} = 2 - kA(t)}$$

   (b) Suppose the constant of proportionality in your differential equation is -1 and that initially you have 100 milliliters of alcohol in your body. Find $A(t)$.

   Comment: The problem statement here is not quite clear when it mentions the constant of proportionality. The intention was that the differential equation should be $dA/dt = 2 - A(t)$ but the statement is a little ambiguous and could be interpreted as saying that the differential equation should be $dA/dt = 2 - (-1)A(t) = 2 + A(t)$. The latter option does not make sense biologically (it would imply that instead of metabolizing alcohol, your body is somehow *creating* alcohol at a rate proportional to the amount already in your body), but the problem should have been written more clearly. Also, note that in real life the constant of proportionality is much smaller (i.e. people do not metabolize alcohol nearly as quickly as this problem implies).

   Solution: Based on the information in this part of the problem, we have

   $$\frac{dA}{dt} = 2 - A(t);\ A(0) = 100.$$

   To solve this we can use either separation of variables or an integrating factor. To use an integrating factor, we first rearrange the equation to get

   $$A' + A = 2.$$

   The integrating factor is

   $$I(t) = e^{\int dt} = e^t.$$

   Multiplying both sides of the differential equation by this factor gives us

   $$\begin{aligned} I(t)A'(t) + I(t)A(t) &= 2I(t) \\ \implies \frac{d}{dt}(I(t)A(t)) &= 2e^t \end{aligned}$$

   Integrating both sides gives us

   $$I(t)A(t) = \int 2e^t\,dt = 2e^t + C$$

where C is some constant. Dividing by $I(t)$ gives us

$$A(t) = \frac{2e^t + C}{e^t} = 2 + Ce^{-t}.$$

Finally, we need to use the initial value to find what C should be. We have

$$100 = A(0) = 2 + Ce^{-0} = 2 + C$$

and so $C = 98$. Thus the final answer is $\boxed{A(t) = 2 + 98e^{-t}}$.

(c) With the same assumptions as in part (b), what is the amount of alcohol in your body as $t \to \infty$?

Solution: This is simply asking us to find the limit of $A(t)$ as $t \to \infty$.

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} 2 + 98e^{-t} = \boxed{2 \text{ mL}}.$$

2. (2 points) Consider the recurrence relation $a_n = -a_{n-1} + 5a_{n-2}$ with initial conditions $a_0 = 3$ and $a_1 = 4$. True or false: the formula $a_n = n^3 + 3$ is a solution.

○ True √ **False**

Solution: First we check the initial conditions:

$$\begin{aligned} a_0 &= 0^3 + 3 = 3 \\ a_1 &= 1^3 + 3 = 4. \end{aligned}$$

Now we check the recurrence relation:

$$\begin{aligned} -a_{n-1} + 5a_{n-2} &= -((n-1)^3 + 3) + 5((n-2)^3 + 3) \\ &= -(n^3 - 3n^2 + 3n - 1 + 3) + 5(n^3 - 6n^2 + 12n - 8 + 3) \\ &= 4n^3 - 27n^2 + 57n - 27 \end{aligned}$$

It seems implausible that this is equal to $a_n = n^3 + 3$ for all n. To see concretely that it's not, try plugging in $n = 3$:

$$3^3 + 3 = 30 \neq 9 = 4(3)^3 - 27(3)^2 + 57(3) - 27.$$

3. (2 points) The differential equation $y' = y + ye^t$ is separable.

√ **True** ○ False

Solution: We can rewrite it as $y' = y \cdot (1 + e^t)$, which is a product of something that depends only on y and something that depends only on t.

4. (2 points) Have a good spring break. (Hint: both answers are correct)

√ **True**

○ False

Solution: :)