Math 10B, Quiz 7 Solutions

1. (12 points) Suppose you roll a fair six-sided die repeatedly until you get 20 threes. Define the random variable X to be the number of times you had to roll the die. What is $E[X]$? (Hint: try writing X as a sum of random variables whose expected values you can figure out.)

Solution: Define random variables $X_1, X_2, \ldots, X_{20}$ as follows:

- X_1 is the number of rolls before the first 3.
- X_2 is the number of rolls after the first 3 but before the second 3.
- X_3 is the number of rolls after the second 3 but before the third 3.
- $\ldots$
- X_{20} is the number of rolls after the nineteenth 3 but before the twentieth 3.

So $X = X_1 + X_2 + \ldots + X_{20} + 20$ (the $+20$ is to account for the number of times 3 is rolled, since these rolls are not counted in any of the X_i's). By linearity of expectation,

$$E[X] = E[X_1] + \ldots + E[X_{20}] + 20.$$

Each X_i is a geometrically-distributed random variable with parameter $\frac{1}{6}$ (the chance of rolling a 3) so

$$E[X_i] = \frac{1-\frac{1}{6}}{\frac{1}{6}} = 5.$$

So $\boxed{E[X] = 20 \cdot 5 + 20 = 120}$.

2. (1 point) If X and Y are random variables then $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$. ○ True √ **False**

Solution: This is true when X and Y are independent, but not true in general. For instance, suppose that $Y = 1 - X$ and X is not constant. Then $X + Y = 1$ so $\mathrm{Var}[X+Y] = 0$ but $\mathrm{Var}[X]$ and $\mathrm{Var}[Y]$ are both positive.

3. (1 point) Let X be a Poisson random variable with parameter $\lambda = 0.2$. Then for any n and $k > 0$, $P(X = n + k \mid X \geq n) = \frac{P(X=n+k)}{P(X \geq n)}$. √ **True** ○ False

Solution: This question actually has nothing to do with the Poisson distribution. From the definition of conditional probability we have

$$P(X = n + k \mid X \geq n) = \frac{P(X = n + k \text{ and } X \geq n)}{P(X \geq n)}.$$

Now observe that since k is nonnegative, if $X = n + k$ then X must also be greater than or equal to n. In other words, the event that $X = n + k$ and $X \geq n$ is the same as the event that $X = n + k$. So $P(X = n + k \text{ and } X \geq n) = P(X = n + k)$.

4. (1 point) On an exam, a question asks, “Suppose you draw 10 cards from a deck. What is the expected number of spades that you get?” One student gives an answer of $10 \cdot \frac{13}{52} = 2.5$ reasoning as follows: we can

think of the number of spades as a random variable following the binomial distribution. The probability of success (i.e. the probability of getting a spade) when drawing one card is $\frac{13}{52}$, there are 10 trials (i.e. 10 cards are drawn) and the expected value of binomially distributed random variables is the number of trials times the probability of success.

- ○ The student's answer is correct and their reasoning is valid.
- ✓ **The student's answer is correct but their reasoning is not valid.**
- ○ The student's answer is incorrect.

Solution: The binomial distribution is the number of successes in a series of trials which are all independent. But the events of getting a spade on the first draw and getting a spade on the second draw are not independent (if you get a spade on your first draw, you are slightly less likely to get one on the second draw). So the student's reasoning is not valid (and if the question instead asked for something like the probability of getting 5 spades then the student's reasoning would cause them to get an incorrect answer).

Instead, this problem is an example of the hypergeometric distribution, which happens to give the same expected value as the one calculated using invalid reasoning by the student.