Math 10B, Quiz 6 Solutions

1. (12 points) Suppose you are investigating the theft of a valuable trading card. There are two suspects, Tom and Sue. Before seeing any evidence, you believe there is a 60% chance that Tom is guilty and a 10% chance that Sue is guilty. You also believe that if Tom committed the crime, there is a 10% chance he would have sold the card on ebay and if Sue committed the crime, there is a 50% chance she would have sold the card on ebay. If you learn that the card was sold on ebay (and no other new evidence) who should you now believe is more likely to be guilty? Make sure to show your work and give a clear explanation of your answer. (Hint: Bayes' theorem. Hint 2: You don't need to know the overall probability that the card would be sold on ebay to answer the question.)

Solution: Let's start by naming the relevant events: let T denote the event that Tom is guily, S the event that Sue is guilty, and E the event that the card was sold on ebay. To know whether Tom or Sue is more likely to be guilty given that the card was sold on ebay, we need to compare $P(T \mid E)$ and $P(S \mid E)$.

First, let's summarize the information we are given:

- $P(T) = 0.6$
- $P(S) = 0.1$
- $P(E \mid T) = 0.1$
- $P(E \mid S) = 0.5$

We can try to calculate $P(T \mid E)$ and $P(S \mid E)$ using Bayes' theorem:

$$P(T \mid E) = \frac{P(E \mid T)P(T)}{P(E)} = \frac{0.1 \cdot 0.6}{P(E)}$$
$$P(S \mid E) = \frac{P(E \mid S)P(S)}{P(E)} = \frac{0.5 \cdot 0.1}{P(E)}$$

The only problem now is that we don't know $P(E)$. And even worse, it's actually impossible to determine $P(E)$ from the information given in the problem since we don't know the probability of the card being sold on ebay if neither Tom nor Sue is guilty (and we probably shouldn't even assume that T and S are disjoint—it is possible that both Tom *and* Sue are guilty). However, it actually doesn't matter that we don't know $P(E)$. No matter what $P(E)$ is, $\frac{0.1 \cdot 0.6}{P(E)}$ is greater than $\frac{0.5 \cdot 0.1}{P(E)}$, so $\boxed{\text{Tom}}$ is more likely to be guilty.

2. (1 point) Suppose A and B are independent events and that $P(A) = 0.5$ and $P(B) = 0.8$. Then $P(A \mid B) = 0.5$. $\surd$ **True** ◯ False

Solution: This is a simple consequence of the definitions of conditional probability and independence: if A and B are independent then

$$\begin{aligned} P(A \mid B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A)P(B)}{P(B)} \\ &= P(A) \end{aligned}$$

3. (1 point) Suppose you roll a fair six-sided die. The following are independent events: rolling a one and rolling a three. ◯ True √ **False**

Solution: The probability of rolling a one is 1/6, the probability of rolling a three is 1/6, and the probability of rolling both is 0 (it is impossible to roll two different numbers on a single roll). Since $0 \neq (1/6) \cdot (1/6)$, the two events are not independent.

4. (1 point) On an exam, a question asks: "You draw four cards from a standard deck of 52. What is the probability of getting no hearts?" One student gives the answer of $1 - \frac{\binom{13}{4}}{\binom{52}{4}}$ reasoning as follows: to find the probability of the event of getting no hearts, you can instead find the probability of its complement and subtract that from 1. The complement of the event of getting no hearts is the event of getting all hearts and the probability of this is $\binom{13}{4}$ (the number of ways to pick 4 hearts) divided by $\binom{52}{4}$ (the number of ways to pick 4 cards of any kind). The student's answer is:

◯ Too small

◯ Correct

√ **Too large**

Solution: The student erred when they said that the complement of getting no hearts is getting all hearts. Instead, the complement is getting *at least one* heart. Since the probability of getting at least one heart is larger than the probability of getting all hearts, the student should have subtracted a larger number from one, and thus should have ended up with a smaller final answer. So the student's answer is too large.

Common Mistakes: Several people appeared to understand the main point of this problem, but forgot that if you subtract a larger number then you get a smaller final answer.