Math 10B, Quiz 4

1. (12 points) A coffee shop sells five sizes of coffee. How many ways are there to order 10 coffees? Make sure to clearly explain your answer.

> **Solution:** This was similar to several homework problems. We can think of the five sizes of coffee as 5 distinguishable boxes and the 10 coffees we must order as indistinguishable balls. To throw a ball into a box means to order one more of that size of coffee (so every ball must go into some box). The balls are indistinguishable because all that matters is how many of each size we order and not which order we order them in. There are no restrictions on how many of each size we order, so this problem can be solved using the stars and bars method. So the solution is
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> $$\binom{10+5-1}{10}$$

2. (1 point) There are more ways to put 10 distinguishable balls into 5 indistinguishable boxes so that each box has at most one ball than there are to put 5 distinguishable balls into 5 indistinguishable boxes so that each box has at most one ball. ○ True $\surd$ **False**

> **Solution:** There are 0 ways to put 10 balls into 5 boxes so that each box has at most one ball—it doesn't even matter whether the balls and boxes are distinguishable or not since there are not enough boxes. This is already enough to know the answer is 'false' since no matter what the second number is, it can't be less than 0. Anyway, there is 1 way to put 5 distinguishable balls into 5 indistinguishable boxes since each ball can be put into a different box and there is no way to tell apart two different ways of doing since the boxes look the same.

3. (1 point) Recall that $S(n,k)$ is the number of ways to put n distinguishable balls into k indistinguishable boxes when every box must have at least one ball. True or false: $S(n+1,k+1) = S(n,k+1)+kS(n,k)$. (It is not necessary to remember the formula from class to solve this problem. Try doing small examples, thinking about what each term means combinatorially, etc.)

   ○ True $\surd$ **False**

> **Solution:** The correct formula is $S(n+1,k+1) = S(n,k) + (k+1)S(n,k+1)$. One way to see it is to think about what happens to the first of the $n+1$ balls: either it is in a box by itself—and there are $S(n,k)$ ways this can happen—or it can be in a box with other balls—and there are $(k+1)S(n,k+1)$ ways this can heppen.
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> However, it was not necessary in this case to know the right formula in order to solve the question. If you pick values of k and n small enough, it is easy to find $S(n,k)$ by hand and then see that the identity does not hold. For instance, $S(3,2) = 4$, $S(2,2) = 2$, and $S(2,1) = 1$ but $2 + 1 \cdot 1 \neq 4$.

4. (1 point) On an exam, a question asks whether the following statement is true: "If there are 32 books on a bookshelf and 7 are marked then there must be two marked books with less than three books in between them." The student claims that the statement is true, reasoning as follows: consider the 32 books as objects and the 7 marked books as boxes. Since $\lceil \frac{32}{7} \rceil > 1$, the pigeonhole principle says that there are two marked books with less than 3 books in between.

   ○ The statement is correct and the student's reasoning is valid.

   ○ The statement is correct but the student's reasoning is not valid.

$\surd$ **The statement is not correct and the student's reasoning is not valid.**

Solution: The statement is incorrect because there are ways to mark 7 of the books so that no two marked books have less than three books in between them. In fact, there are many such ways. One of them is to mark the books 1, 5, 10, 15, 20, 25, and 30, but this is not the only way.

Since the statement is incorrect, the student's reasoning had better not be valid. The problem is that the student wants to use the pigeonhole principle, but leaves out two important things: how the objects are assigned to the boxes, and how some box having more than one object in it implies that there are two marked books with less than three books between them. Without these two components, the pigeonhole principle does not tell us anything.

The lesson here is to beware of trying to apply the pigeonhole principle by simply finding two numbers in the problem and dividing them. Doing this can sometimes lead you to make false statements.