Math 10B, Quiz 3

1. (12 points) Suppose 200 students in math 10B this semester were all born in the same year. Show that there are at least two students who either have the same birthday or were born on consecutive days.

Comment: Some people assumed that the year on which the 200 students were born was a leap year and some people assumed it was not a leap year. It doesn't make a big difference either way, so I will assume it is not a leap year (and since most students in 10B this semester were probably born after 1996 and before 2004, this is a pretty reasonable assumption).

Solution: Divide the days of the year into groups of two—e.g. {January 1, January 2}, {January 3, January 4}, ... , {December 29, December 30}, {December 31} (the last group has just one day in it because there are an odd number of days in the year).

Consider the 200 students as objects and the $\lceil \frac{365}{2} \rceil = 183$ pairs of days as boxes and put each student in the box that includes their birthday. Since there are more students than boxes, there must be some box with at least two students. And two students in the same box must either share a birthday or have been born on consecutive days.

2. (1 point) For all n and all $k \leq n$,

$$\binom{n}{k}\binom{n+1}{k-1} = \binom{n}{k-1}\binom{n+1}{k}.$$

○ True √ **False**

Solution: One way to discover that the equality is false is to try some small examples and see if it fails for any of them. For instance, with $n = 3$ and $k = 2$, the left side is $\binom{3}{2}\binom{4}{1} = 12$ and the right side is $\binom{3}{1}\binom{4}{2} = 18$.

Another method is to rewrite both sides in a way that makes them easier to compare. For instance, the left hand side is equal to

$$\frac{n!}{k!(n-k)!} \cdot \frac{(n+1)!}{(k-1)!(n+2-k)!}$$

while the right hand side is equal to

$$\frac{n!}{(k-1)!(n+1-k)!} \cdot \frac{(n+1)!}{k!(n+1-k)!}.$$

Examining both sides, it is possible to see they are never equal.

3. (1 point) For all n, $m \leq n$ and $k \leq m$,

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}.$$

√ **True** ○ False

Solution: If you try doing some small examples here, you will find that both sides keep turning out to be equal. That is not conclusive evidence that they are really always equal, but it is suggestive.

If you have enough time, it is best to try to come up with some proof that they are really always equal. In this case, both algebraic and combinatorial proofs are feasible. For instance, you can show that both sides count the number of ways to choose a team of size m from a pool of n people, where k of the members of the team play defense.

4. (1 point) On an exam, a question asks "How many 6 card hands can you form if the hand has to include at least one spade?" One student gives the answer $13\binom{51}{5}$, reasoning that they need to choose at least one spade (and there are 13 spades to choose from) and then they can choose any 5 of the 51 cards that haven't been used yet. The student is:

   ◯ Undercounting (i.e. their answer is too small)

   ◯ Correct

   √ **Overcounting (i.e. their answer is too large)**

Solution: The students solution is overcounting hands with more than one spade. It is easiest to see this via an example. If we follow the student's reasoning, we might first choose the ace of spades as our spade and then just happen to choose the 2 of spades and the 3, 4, 5, and 6 of hearts as our other 5 cards. Or we instead might first choose the 2 of spades as our spade and then the ace of spades and the 3, 4, 5, and 6 of hearts as our 5 other cards. But these give us the same hand.

Comment: If you have trouble deciding if your solution may be overcounting (or undercounting), or if you have trouble coming up with a solution in the first place, it can often be helpful to first change the problem to one with small enough numbers that you can actually list out all the possibilities. For instance, in this problem you might try a deck of size 4, with two suits and two cards of each suit, and hands of size 3.