# Math 10B, Quiz 10 Solutions

1. (9 points) Solve the following differential equation

$$\frac{t^2A' - 100A'}{A} = 1.$$

**Solution:** There are two methods to solve this problem.

**Method 1: Separation of variables.**

$$\begin{aligned}
\frac{t^2A' - 100A'}{A} &= 1 \\
\frac{A'(t^2-100)}{A} &= 1 && \text{(factor out } A'\text{)} \\
\frac{A'}{A} &= \frac{1}{t^2-100} && \text{(divide both sides by } t^2-100\text{)} \\
\int \frac{dA}{A} &= \int \frac{dt}{t^2-100} && \text{(integrate both sides)} \\
\ln(A) &= \int \frac{dt}{t^2-100} \\
A(t) &= e^{\int \frac{dt}{t^2-100}}
\end{aligned}$$

**Method 2: Integrating factor.** First we need to put the differential equation into a form where we can use an integrating factor.

$$\begin{aligned}
\frac{A'(t^2-100)}{A} &= 1 \\
A' &= \frac{A}{t^2-100} && \left(\text{multiply both sides by } \frac{A}{t^2-100}\right) \\
A' - \frac{1}{t^2-100} &= 0.
\end{aligned}$$

Now that the equation is in the right form, we can see that the integrating factor is

$$I(t) = e^{-\int \frac{dt}{t^2-100}}.$$

Multiplying both sides by the integrating factor gives us

$$\begin{aligned}
I(t)A'(t) - \frac{1}{t^2-100}A(t)I(t) &= 0 \cdot I(t) \\
\implies \frac{d}{dt}(A(t)I(t)) &= 0
\end{aligned}$$

Integrating both sides gives us

$$A(t)I(t) = \int 0 \cdot dt = C$$

where $C$ is some constant. Dividing by $I(t)$ gives $A(t)$.

$$A(t) = \frac{C}{I(t)} = \frac{C}{e^{-\int \frac{dt}{t^2-100}}} = Ce^{\int \frac{dt}{t^2-100}}.$$

In either method, we now need to compute $\int \frac{dt}{t^2-100}$. We can do this using partial fraction decomposition. Specifically, we can first factor $t^2 - 100$ into $(t-10)(t+10)$ and then find constants $a$ and $b$ such that

$$\frac{1}{t^2-100} = \frac{a}{t-10} + \frac{b}{t+10}.$$

Any such $a$ and $b$ must satisfy

$$a(t+10)+b(t-10)=1$$

for all $t$. Plugging in $t=10$ and $t=-10$ gives us $a=1/20$ and $b=-1/20$. Therefore

$$\begin{aligned}\int \frac{dt}{t^2-100} &= \frac{1}{20}\int \frac{dt}{t-10} - \frac{1}{20}\int \frac{dt}{t+10} \\ &= \frac{1}{20}\ln(t-10) - \frac{1}{20}\ln(t+10) + c\end{aligned}$$

where $c$ is some constant.

Plugging this into the formulas for $A(t)$ that we found in either method gives us

$$A(t) = Ce^{\frac{1}{20}\ln(t-10)-\frac{1}{20}\ln(t+10)} = C(t-10)^{1/20}(t+10)^{-1/20} = \boxed{C\left(\frac{t-10}{t+10}\right)^{1/20}}.$$

2. (2 points) Separation of variables can be used to solve $y'' = y' + y$.

○ True √ **False**

**Solution:** The differential equation is not first order, so separation of variables cannot be used.

3. (2 points) $y(t) = \cos(t) + 5$ is a solution to the differential equation $y'(t)\cos(t) + y'(t)y(t) = -5\sin(t)$. (Hint: the derivative of $\cos(t)$ is $-\sin(t)$.)

○ True √ **False**

**Solution:** To check if the given function is a solution to the given differential equation, we just need to plug the function in and see if the equation is satisfied. For the function $y(t)$ given, $y'(t) = -\sin(t)$ so we have

$$y'(t)\cos(t) + y'(t)y(t) = -\sin(t)\cos(t) + -\sin(t)(\cos(t)+5) = -2\sin(t)\cos(t) - 5\sin(t).$$

Since $-2\sin(t)\cos(t)$ is not equal to 0 for all $t$, the expression above is *not* equal to $-5\sin(t)$ and so $y(t)$ is **not** a solution.

4. (2 points) A student is asked to write a differential equation to model the amount of water in a puddle in the following scenario: "A puddle of water initially contains 50 mL of water. Water evaporates from the puddle at a rate proportional to the amount of water in the puddle. There is also a light rain which adds water to the puddle at a rate of 5 mL per minute." The student writes

$$\frac{dW}{dt} = 5t - kW(t);\ W(0) = 50$$

where $W(t)$ is the amount of water in the puddle (in mL) after $t$ minutes and $k$ is a constant. The student's reasoning is as follows: the derivative of $W$ is how much water is entering the puddle minus how much water is leaving the puddle. After $t$ minutes, $5t$ mL of water have entered the puddle and water is leaving the puddle through evaporation at a rate that is some constant multiple of the amount of water in the puddle. Also, at time 0 there are 50 mL of water in the puddle. The student's answer is:

○ Correct with valid reasoning.

○ Correct with invalid reasoning.

√ **Incorrect.**

**Solution:** It is true that the amount of water that has entered the water after $t$ minutes is $5t$ mL, but $\frac{dW}{dt}$ depends on the *rate* at which water enters the puddle, not on the total amount that has entered. The correct differential equation would be

$$\frac{dW}{dt} = 5 - kW(t);\ W(0) = 50.$$