Combinatorics Worksheet 7: Twelvefold Way

1. Suppose you have 8 boxes labelled 1 through 8 and 16 indistinguishable red balls. How many ways are there to put the balls into the boxes if:

   (a) No odd box can be empty.

   Solution: To find all the ways to put the balls into the boxes, we can first put a ball into every odd box—leaving $16 - 4 = 12$ balls—and then distribute the remaining balls into the 8 boxes in any way. At this point, it is just a standard stars and bars style problem with 12 balls and 8 boxes. The answer is $\boxed{\binom{12+8-1}{12}}$.

   (b) Odd boxes must have an odd number of balls and even boxes, an even number of balls.

   Solution: Many people found this one challenging. There are two tricks here: first put one ball into every odd box, as in part (a). Second, after putting one ball into each odd box, group the balls into pairs and decide how many pairs to put into each box. After putting one ball into each odd box, there are 12 balls remaining, or 6 pairs. Finding the number of ways to put the pairs into the boxes is a standard stars and bars style problem with 6 stars (the pairs) and $8-1 = 7$ bars. So the final answer is $\boxed{\binom{6+7}{6}}$.

   (c) You also have 16 indistinguishable green balls and want to distribute both the red and green balls into the boxes.

   Solution: We can think of this as two separate tasks: first put the green balls into the boxes and then put the red balls into the boxes. Since the way we choose to complete the first task does not affect how many ways there are to complete the second task, we can just multiply together the number of ways to do each task on its own. And each task on its own is just a standard stars and bars style problem with 16 stars and $8 - 1 = 7$ bars. So the final answer is $\boxed{\binom{16+7}{16} \cdot \binom{16+7}{16}}$.

2. Draw the twelvefold way chart from class. For each question below, determine which part of the chart it fits into and find the solution (you don't need to find a numerical solution—writing Stirling numbers or partition numbers for some answers is fine).

   Solution:

Balls	Boxes	Any	Injective	Surjective
D	D	(g)	(c)	(h)
I	D	(d)	(e)	(f)
D	I	(a)	(i)	(j)
I	I	(b)		

   (a) How many rhyme schemes are there for a poem with n lines?

Solution:

$$\sum_{j=1}^{n} S(n,j)$$

(b) How many ways are there to write n as a sum of at most 12 positive integers?

Solution:

$$\sum_{j=1}^{12} p_j(n)$$

(c) How many ways are there to form a word of length 10 (doesn't need to be a real english word) with no repeated letters?

Comment: Originally, I forgot to include the "length 10" requirement. It doesn't change the problem much, but it makes it fit more precisely into the twelvefold way chart.

Solution:

$$P(26,10) = \frac{26!}{(26-10)!}$$

(d) If you have 100 hours of work to do and 12 employees, how many ways are there to divide the hours of work between the employees? Assume all the hours are of the same type of work and do not need to be scheduled at specific times.

Solution:

$$\binom{100+12-1}{100}$$

(e) Same as the previous question, but now you have 200 employees and no employee may work more than one hour.

Solution:

$$\binom{200}{100}$$

(f) Same as part (d), but every employee must work at least one hour.

Solution:

$$\binom{100-12+12-1}{100-12} = \binom{100-1}{100-12}$$

(g) How many ways are there to make a password of length 15 containing only digits?

Solution:
$$10^{15}$$

(h) What if the password must include every digit?

Solution:
$$10!S(15,10)$$

(i) If there are 10 identical music practice rooms and 6 students, how many ways are there to assign students to practice rooms (assume that every student should have their own room)?

Solution: There is only $\boxed{1}$ way since the practice rooms are indistinguishable.

(j) How many ways are to divide 20 students into exactly 6 (nonempty) study groups?

Solution:
$$S(20,6)$$

3. Optional question (only do it once you've finished the other questions). Show that when using the version of the stable marriage algorithm in which the men propose to the women, it is possible that every woman gets her last choice.

Solution: The key observation is that if all the men have different top choices then they will all get their top choice (no woman will get more than one proposal on the first round so no woman will be able to reject any proposals). But the men's top choices may not like them very much. For a concrete example, suppose A, B, and C are the men, X, Y, and Z are the women, and their preferences are:

A	B	C	X	Y	Z
X	Y	Z	C	A	B
Y	Z	X	B	C	A
Z	X	Y	A	B	C

If you run the algorithm you will find that every man gets his top choice and every woman her last choice.

Comment: In fact, it is possible to prove that when the men propose, every man gets the best option he could get in any stable matching and every woman gets the worst option she could get in any stable matching.

4. Optional question (only do it once you've finished the other questions). Show that when using the version of the stable marriage algorithm in which the men propose to the women, it is possible that some women have an incentive to lie—i.e. if they lie about their preferences they

can end up with a higher ranked man (according to their true preferences) than the one they would end up with if they did not lie.

Solution: Suppose the men are A, B, and C, the women are X, Y, and Z, and their preferences are:

A	B	C	X	Y	Z
X	X	Y	C	B	A
Y	Y	X	B	C	B
Z	Z	Z	A	A	C

If you run the algorithm with the men proposing, you will find that X gets matched with B. But if on the first round, X lies about her preferences and says she likes A better than B, she will instead end up with C, who is her top choice.

Comment: It turns out that protecting against lying when finding stable matchings is hard to do. In fact, there is no algorithm in which the participants never have an incentive to lie. On the other hand, some people have studied real life applications of the stable marriage algorithm and found that in most cases that come up in real life, nobody has an incentive to lie (basically because preference lists are correlated: if a certain man is well-liked by one woman it is likely that he is well-liked by a lot of other women as well).

In general though, whenever an algorithm is supposed to take into account the input of many participants, it is useful to ask whether any of the participants have an incentive to lie or cheat in some other way. And coming up with algorithms that discourage such cheating is the subject of plenty of current research (especially in the fields of cryptography and mechanism design).

5. **Challenge Problem:** In a sequence of coin tosses, one can keep a record of instances in which a tail is immediately followed by a head, a head is immediately followed by a head, and etc. We denote these by TH, HH, and etc. For example, in the sequence TTTHHTHTTTHHTTH of 15 coin tosses we observe that there are five HH, three HT, two TH, and four TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH, and five TT subsequences?