Combinatorics Worksheet 7: Twelvefold Way

1. Suppose you have 8 boxes labelled 1 through 8 and 16 indistinguishable red balls. How many ways are there to put the balls into the boxes if:
   (a) No odd box can be empty.
   (b) Odd boxes must have an odd number of balls and even boxes, an even number of balls.
   (c) You also have 16 indistinguishable green balls and want to distribute both the red and green balls into the boxes.

2. Draw the twelvefold way chart from class. For each question below, determine which part of the chart it fits into and find the solution (you don't need to find a numerical solution—writing Stirling numbers or partition numbers for some answers is fine).
   (a) How many rhyme schemes are there for a poem with n lines?
   (b) How many ways are there to write n as a sum of at most 12 positive integers?
   (c) How many ways are there to form a word of length 10 (doesn't need to be a real english word) with no repeated letters?
   (d) If you have 100 hours of work to do and 12 employees, how many ways are there to divide the hours of work between the employees? Assume all the hours are of the same type of work and do not need to be scheduled at specific times.
   (e) Same as the previous question, but now you have 200 employees and no employee may work more than one hour.
   (f) Same as part (d), but every employee must work at least one hour.
   (g) How many ways are there to make a password of length 15 containing only digits?
   (h) What if the password must include every digit?
   (i) If there are 10 identical music practice rooms and 6 students, how many ways are there to assign students to practice rooms (assume that every student should have their own room)?
   (j) How many ways are to divide 20 students into exactly 6 (nonempty) study groups?

3. Optional question (only do it once you've finished the other questions). Show that when using the version of the stable marriage algorithm in which the men propose to the women, it is possible that every woman gets her last choice.

4. Optional question (only do it once you've finished the other questions). Show that when using the version of the stable marriage algorithm in which the men propose to the women, it is possible that some women have an incentive to lie—i.e. if they lie about their preferences they can end up with a higher ranked man (according to their true preferences) than the one they would end up with if they did not lie.

5. **Challenge Problem:** In a sequence of coin tosses, one can keep a record of instances in which a tail is immediately followed by a head, a head is immediately followed by a head, and etc. We denote these by TH, HH, and etc. For example, in the sequence TTTHHTHTTTHHTTH of 15 coin tosses we observe that there are five HH, three HT, two TH, and four TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH, and five TT subsequences?