Combinatorics Worksheet 6: Review and 12-Fold Way

1. Suppose that you have n employees and need to choose some of them to receive a promotion. In each of the following scenarios, how many ways are there to choose which employees receive a promotion?

   (a) Suppose you can choose any number of employees to receive a promotion.

   > **Solution:** This is just the number of subsets of a set of size n, so the answer is $\boxed{2^n}$. The problem does not say that it is forbidden to not give a promotion to anybody, so we don't need to worry about excluding the empty set.

   (b) Exactly 5 employees must receive a promotion.

   > **Solution:** This is just the number of 5-combinations of a set of size n, or $\boxed{\binom{n}{5}}$.

   (c) Any number of employees can receive a promotion, but at least one of the employees Alan, Kim, and Cassandra must receive a promotion.

   > **Solution:** There are several ways to do this. One is to count the number of ways to choose which employees receive a promotion when none of Alan, Kim, or Cassandra can receive a promotion, and then subtract this number from the total number of ways to choose which employees receive promotions when there are no restrictions. If Alan, Kim, and Cassandra cannot receive promotions then we just need to pick which of the $n-3$ other employees receive promotions. There are 2^{n-3} ways to do this. So the final answer is $\boxed{2^n - 2^{n-3}}$.
   > Another way to solve the problem is to find the number of ways to choose at least one of Alan, Kim, and Cassandra and multiply that by the number of ways to choose any number of the other $n-3$ employees to receive a promotion. There are 2^3-1 ways to do the former (the number of subsets of $\{\text{Alan}, \text{Kim}, \text{Cassandra}\}$ besides the empty set) and there are 2^{n-3} ways to do the latter. So the final answer is $(2^3-1)2^{n-3}$.

   (d) Any number of employees can receive a promotion, but at *most* one of the employees Alan, Kim, and Cassandra must receive a promotion.

   > **Solution:** We can solve this problem using a method similar to the second one in the solution to the previous part. Namely, first find the number of ways to choose at most one out of Alan, Kim, and Cassandra, and then multiply that by the number of ways to choose any number of the other $n-3$ employees. There are $\binom{3}{1} + \binom{3}{0} = 3 + 1 = 4$ ways to choose at most one of Alan, Kim, and Cassandra and 2^{n-3} ways to choose any number of the other employees. So the answer is $\boxed{4 \cdot 2^{n-3}}$.

2. What is the coefficient of x^6y^7 in $(3x^2 - y)^{10}$?

Solution: It might be helpful to think of this question as asking for the coefficient of $(x^2)^3y^7$. By the binomial theorem, the answer is $\boxed{\binom{10}{7}(3)^3(-1)^7}$.

3. Give a combinatorial proof of the fact that for all n and all $k \leq n$,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Solution: We will show that both sides of the claimed equality count the number of ways to choose k people from a class of $n+1$ students, one of whom is named Bob.

One way to count this is $\binom{n+1}{k}$: the number of ways to choose k things out of $n+1$ things.

Another way to count it is to break it into two cases: first count the number of ways to choose k people where Bob is one of the k and then count the number of ways to choose k people where Bob is not one of the k. There are $\binom{n}{k-1}$ ways to do the former—we just have to choose the other $k-1$ people from the rest of the class (besides Bob, who we already picked). There are *binomnk* ways to do the latter—if we are not allowed to use Bob then we have to choose all k people from the other n students. So the total number of ways to choose k people is the sum of the two cases: $\binom{n}{k-1} + \binom{n}{k}$.

4. How many ways are there to pay your employees if you have \$1000 and 5 employees? Assume that you are allowed to pay employees nothing and that you don't have to spend all \$1000. Also assume that you must pay employees in dollar amounts (e.g. you cannot pay someone \$4.53).

Solution: We can view the employees as distinguishable boxes and the dollars as indistinguishable balls. The only problem is that not all the balls are required to go into a box—i.e. not all the dollars have to be paid to some employee. One way to deal with this is to add an extra box for dollars that don't go to any employee. So we will have 6 distinguishable boxes and 1000 indistinguishable balls. The problem can now be solved using stars and bars, giving $\boxed{\binom{1000+6-1}{1000}}$.

5. How many ways are there to arrange 20 books on a bookcase with 3 shelves? Assume, as in real life, that books are distinguishable and that the order of the books on each shelf matters.

Solution: To put 20 books on 3 shelves, you can first put the books in order and then decide how many go on each shelf. There are $P(20, 20) = 20!$ ways to put the books in order and, by stars and bars, $\binom{20+3-1}{20}$ ways to choose how many books go on each shelf. So the final answer is $\boxed{20!\binom{22}{20}}$.

6. **Challenge Problem:** Find as many interesting patterns as possible in Pascal's triangle.